MATHEMATICS 31

MODULE 6

Applications of the Derivative

Distance Learning
Alberta EDUCATION

Mathematics 31

Module 6

APPLICATIONS OF THE DERIVATIVE

Distance Learning

Alberta EDUCATION

This document is intended for	
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Teachers (Mathematics 31)	✓
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Parents	
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Mathematics 31
Student Module Booklet
Module 6
Applications of the Derivative
Alberta Distance Learning Centre
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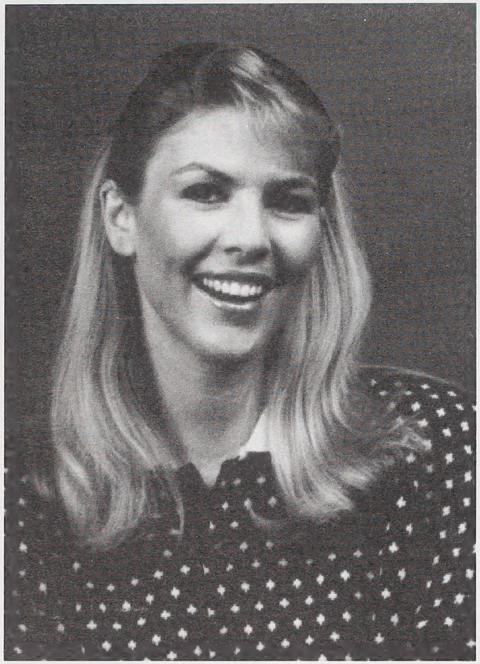
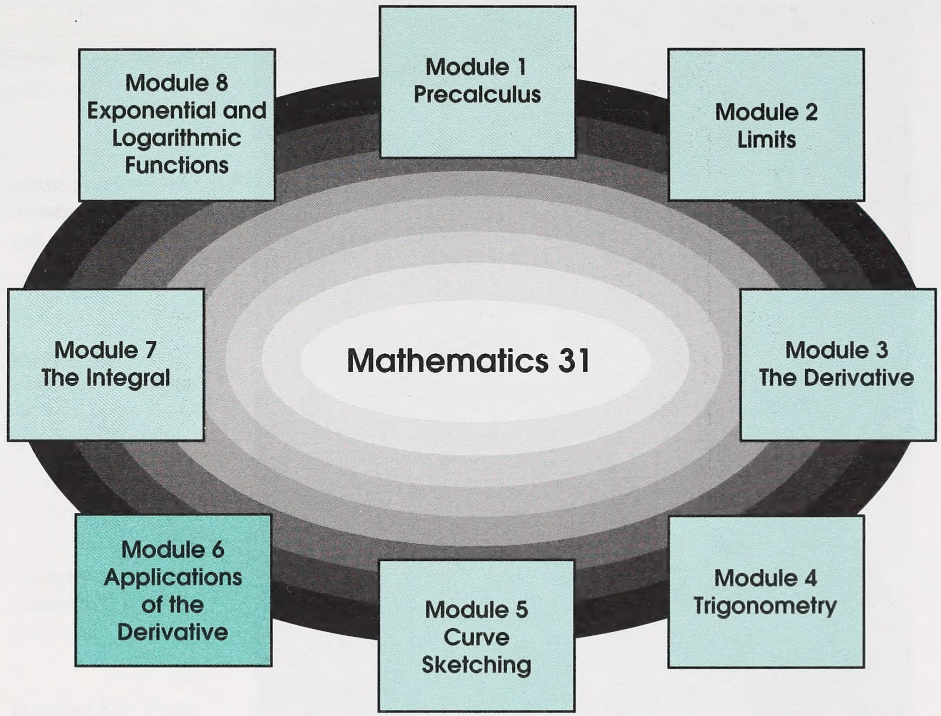
Welcome



WESTFILE INC.

Welcome to Module 6. We hope you'll enjoy your study of Applications of the Derivative.

Mathematics 31 contains eight modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



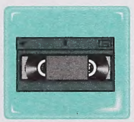
The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.



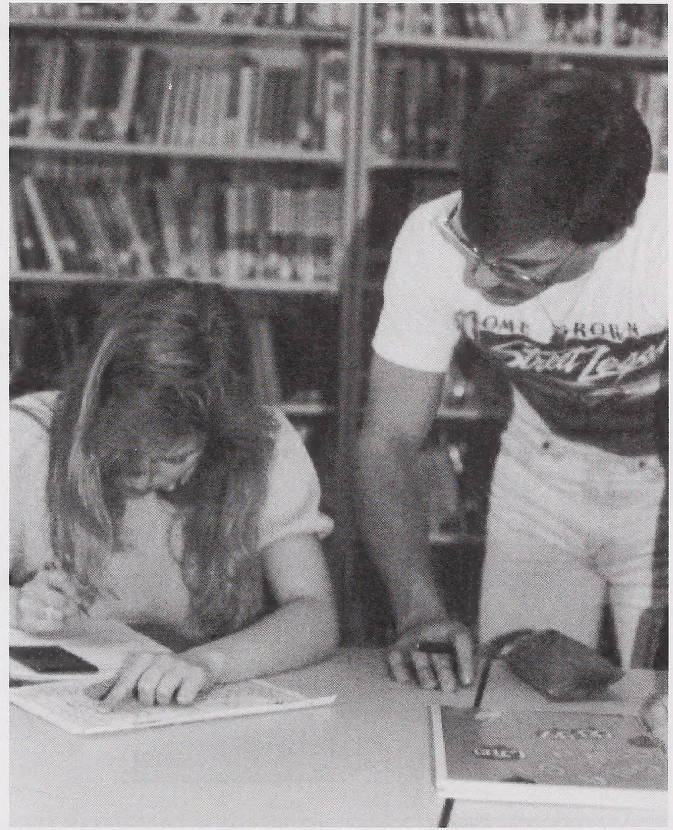
- View a videocassette.

- Pay close attention to important words or ideas.

Assignment Booklet

- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

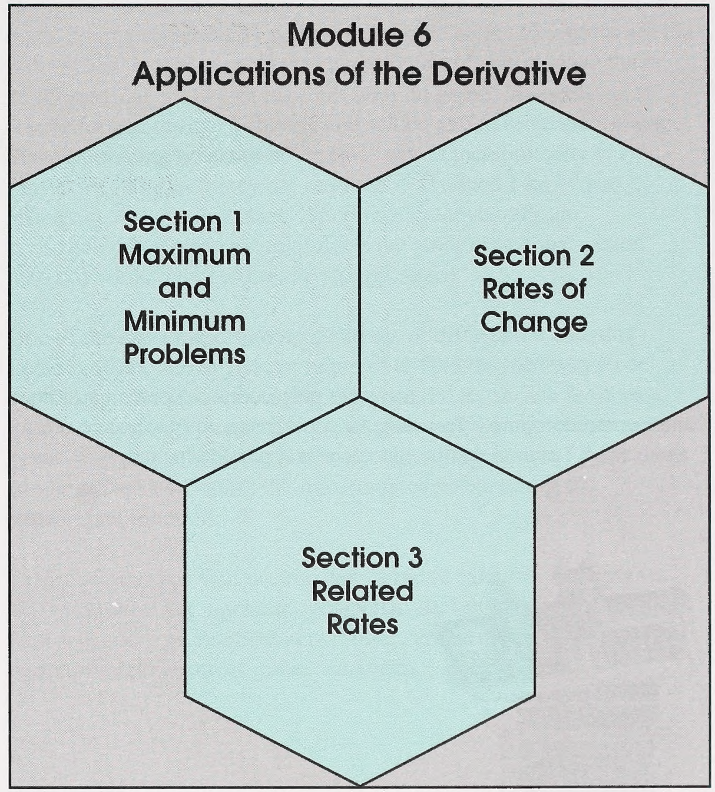
Module Overview

How do you construct a container of maximum volume with a minimum amount of material? How can you maximize profit while minimizing costs in a business venture? How would you determine the velocity and acceleration of a moving object? How could you calculate the rate of change of a property or quantity? These are questions calculus can help answer.

It is not by chance that the individual cells of a honeycomb are hexagonal in shape. The minimum material, and therefore the least energy the bees must expend, is required to construct a comb of this shape. Calculus can be used to help explain the way nature conserves material and energy.

In the last module you were shown how to find the derivatives of many different functions. These methods for finding extreme values have practical applications in many areas of life. In this module you will discover how derivatives are used in engineering, business, and in physical, biological, and social sciences to answer some of the previous questions.

As you can see, there are numerous practical applications for calculus. As you work through this module you will find many examples of these applications.



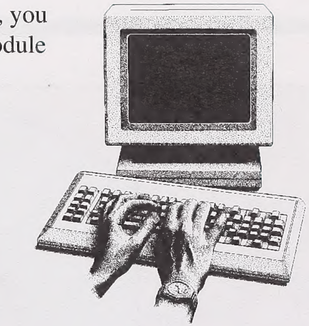
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete three section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**24 marks**
Section 3 Assignment	**21 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Maximum and Minimum Problems

ALBERTA AGRICULTURE FOOD AND RURAL DEVELOPMENT

Honeybees build their honeycombs from pure wax. The combs are made up of geometrically perfect hexagonal cells. Hexagons are the most efficient way to subdivide a given unit of space, since they tessellate (that is, they fit together with no gaps). Bees use these combs for the storage of honey and pollen and to raise their young. Since producing wax and building combs is metabolically very expensive (8 kg of honey are required to produce 1 kg of wax), efficiency is very important. Therefore, honeycombs are constructed in a way that minimizes the amount of construction material while maximizing the storage space.

One of the most important applications of differential calculus is **optimization**, where you are required to find the best way to do something. You have probably encountered terms like *least cost*, *greatest profit*, *optimum size*, *greatest strength*, *least time*, or *greatest distance*, which relate to optimization. Most of these cases can be solved by finding the maximum or minimum of the appropriate function.

In this section you will use the derivative to calculate maximums and minimums for algebraic, geometric, and economic applications. You will solve problems that maximize areas, volumes, and profits, and minimize distances, times, and costs.

Activity 1: Number Problems

Different pairs of numbers may have the same sum but they will have different products. Among these products it is possible to find a maximum value. Trial and error could be used to identify these values, but that procedure would become time-consuming as the given sum increases. The following example shows a way that calculus solves this and similar problems.

Example 1

Find two positive numbers whose sum is 15 and whose product is a maximum.

Solution

Let x and y be the two numbers, and P be their product.

The statement of the problem yields two equations.

$$x + y = 15 \qquad xy = P$$

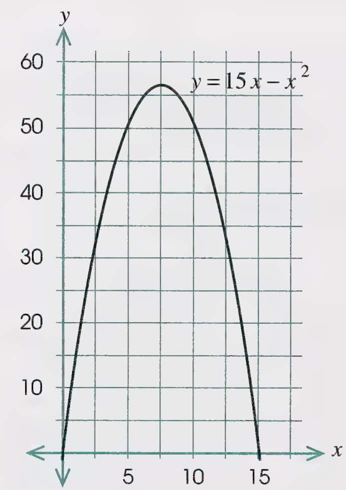
The product $xy = P$ must be maximized, but first P should be expressed in terms of one variable. The condition $x + y = 15$ can be written so that y is in terms of x.

$$y = 15 - x$$

Substitute this into the equation for P.

$$\begin{aligned} P &= x(15 - x) \\ &= 15x - x^2 \end{aligned}$$

When you look at the graph of this function, notice that there is a maximum. Remember from your study of curve sketching, a maximum or minimum occurs when the slope of the function at a particular point is zero.



The task now is to find the critical point where the slope is zero.

Find the derivative of P.

$$\frac{dP}{dx} = 15 - 2x$$

To find the critical numbers, equate the derivative to 0.

$$\begin{aligned} 15-2x&=0 \\ 2x&=15 \\ x&=7.5 \end{aligned} \qquad \begin{aligned} \therefore\ y&=15-x \\ &=7.5 \end{aligned}$$

Verify

This can be done in two ways:

Method 1: Use the First Derivative Test

For $0<x<7.5$, $\frac{dP}{dx}>0$; and for $x>7.5$, $\frac{dP}{dx}<0$.

Therefore, P has an absolute maximum at 7.5 using the First Derivative Test.

Method 2: Use the Second Derivative Test

$$\frac{dP}{dx}=15-2x$$

$$\frac{d^2P}{dx^2}=-2$$

Since the second derivative is negative, the curve is concave downward and there is a maximum at $x=7.5$.

Example 2

Find two positive numbers whose product is 108 and the sum of the first number plus three times the second number is a minimum.

Solution

Let x and y be the two numbers, and S be their sum.

The two equations are as follows:

- $xy=108$
- $x+3y=S$

To find S, substitute $x=\frac{108}{y}$ into $x+3y=S$.

$$S=\frac{108}{y}+3y$$

Take the derivative of the function, and set it equal to zero to find a critical number.

$$\begin{aligned} \frac{dS}{dy}=-\frac{108}{y^2}+3&=0 \\ 3&=\frac{108}{y^2} \\ y^2&=36 \\ y&=6 \end{aligned}$$

Remember, the problem says positive numbers.

This is a minimum value because the second derivative $\left(\frac{d^2S}{dy^2} = \frac{216}{y^3}\right)$ is positive when $y = 6$.

If $y = 6$, then $x = \frac{108}{6}$

$= 18$

Therefore, the two required numbers are 6 and 18.

The following questions give you a chance to practise this concept.

1. The sum of two natural numbers is 28. Find these two numbers if their product is a maximum.

2. Find two consecutive natural numbers if the sum of the larger number and four times the reciprocal of the smaller number is a minimum.

3. The sum of two positive numbers is 4. If the sum of their cubes is a minimum, what are these numbers?

4. Find two numbers that satisfy the conditions that the sum of the first number and twice the second number is 100, and the product is a maximum.

Check your answers by turning to the Appendix.

Next, you will apply the procedures for solving number problems to problems involving geometric figures.

Activity 2: Geometric Applications

How would you enclose a rectangular plot of land with a given amount of fencing in order to maximize the plot's area, or construct a box of greatest volume given a specific amount of material? These problems could be solved by trial and error; but, again, calculus provides a shortcut.

Example 1

Find the dimensions of a rectangle if the perimeter is 28 cm and the area is a maximum. What is the maximum area?

Solution

Determine the width w in terms of length ℓ.

$$\begin{aligned} P &= 2\ell + 2w \\ 28 &= 2(\ell + w) \\ 14 &= \ell + w \\ w &= 14 - \ell \end{aligned}$$

Let A be the maximum area. Write the area in terms of one variable.

$$\begin{aligned} A &= \ell \times w \\ &= \ell(14 - \ell) \\ &= 14\ell - \ell^2 \end{aligned}$$

To find the maximum area, take the derivative of the function and equate it to zero.

$$\frac{dA}{dx} = 14 - 2\ell = 0$$
$$2\ell = 14$$
$$\ell = 7$$

$$\therefore\ w = 14 - \ell$$
$$= 14 - 7$$
$$= 7$$

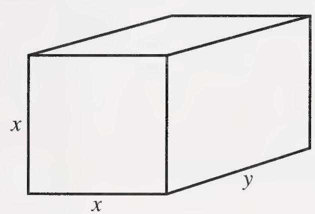
The second derivative $\frac{d^2A}{dx} = -2$. This indicates a maximum. Thus, the dimensions are 7 cm by 7 cm. The maximum area is $49\ \text{cm}^2$.

Example 2

A rectangular box with two squares ends has a total surface area of $150\ \text{cm}^2$. Find the dimensions of the box if the volume is a maximum. What is the maximum volume?

Solution

Let x^2 be the area of a square end. Let xy be the area of a side. Let V be the volume to be maximized.



$$\text{Surface Area} = 2x^2 + 4xy$$
$$150 = 2\left(x^2 + 2xy\right)$$
$$75 = x^2 + 2xy$$
$$2xy = 75 - x^2$$
$$y = \frac{75 - x^2}{2x}$$

The volume formula can be written in terms of one variable.

$$V = x^2 y$$
$$= x^2\left(\frac{75 - x^2}{2x}\right)$$
$$= \frac{75x}{2} - \frac{1}{2}x^3$$

The derivative of the volume, equated to zero, will yield a critical value for x.

$$\frac{dV}{dx} = \frac{75}{2} - \frac{3}{2}x^2 = 0$$
$$\frac{3}{2}x^2 = \frac{75}{2}$$
$$x^2 = 25$$
$$x = 5$$

Since x represents length, it cannot be negative.

$$\text{When } x=5,\ y=\frac{75-5^2}{2(5)}$$
$$=\frac{50}{10}$$
$$=5$$

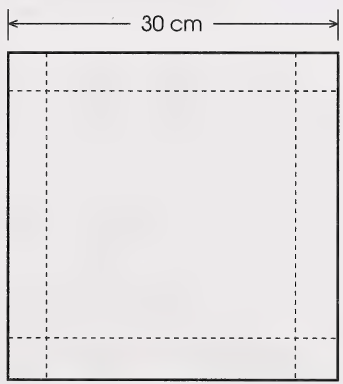
$$\text{At } x=5,\ \frac{d^2V}{dx^2}=-3x$$
$$=-3(5)$$
$$=-15$$

Thus, $x=5$ is a maximum. Therefore, the dimensions of the box are 5 cm by 5 cm by 5cm.

The maximum volume is $5\times5\times5=125\ \text{cm}^3$.

Example 3

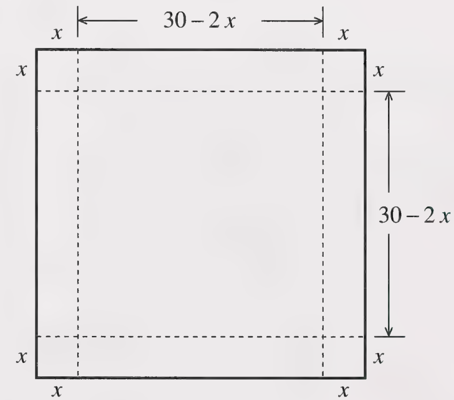
An open box is to be made from a square piece of material, 30 cm on a side, by cutting equal squares from each corner and turning up the sides. Find the volume of the largest box that can be made this way.



Solution

Let x be the side (in cm) of a small square (which becomes the height of the box) and $30-2x$ be the length (in cm) of the new side (which becomes the length and width).

Let V be the volume to be maximized.



$$V=x(30-2x)(30-2x)$$
$$=900x-120x^2+4x^3$$

To find a critical value, equate the derivative to zero.

$$\frac{dV}{dx} = 900 - 240x + 12x^2$$

$$12x^2 - 240x + 900 = 0$$

$$x^2 - 20x + 75 = 0$$

$$(x-5)(x-15) = 0$$

$$x - 5 = 0 \quad \text{or} \quad x - 15 = 0$$

$$x = 5 \qquad\qquad x = 15$$

The solution $x = 15$ is not valid if the length and width are represented by $30 - 2x$. Therefore, the dimensions of the box are 20 cm by 20 cm by 5 cm. The maximum volume is $20 \times 20 \times 5 = 2000 \text{ cm}^3$.

If you have access to a computer and the software *The Geometer's Sketchpad*, load the sketch entitled *Box Volume*. Follow the instructions given on the screen to produce a visual demonstration of maximizing volume.

Try these problems.

1. Find the dimensions of a rectangle if the area of the rectangle is 324 cm^2 and the perimeter is a minimum.

2. Find the dimensions of a rectangle if the perimeter is 100 cm and the area of the rectangle is a maximum.

3. A rectangular box which is open at the top has a square base. The volume is 171.5 cm^3. If the box is to require the least amount of material, what must its dimensions be?

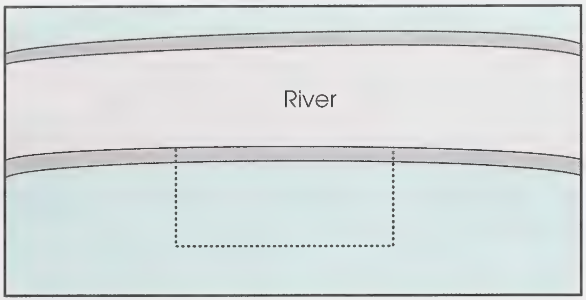
4. A can has a circular base. It is open at the top and is in the form of a circular cylinder. It has a volume of $8000\pi \text{ cm}^3$. If the can is to require the least amount of material, what must its dimensions be?

5. Show that a rectangle with a given area has a minimum perimeter when it is a square.

Check your answers by turning to the Appendix.

Example 5

A farmer wishes to fence part of a rectangular field along a straight river as shown in the following diagram. It is not necessary to fence the side bordering the river. The area of the rectangular field is to be 1800 m^2 and the farmer wishes to use the least length of fencing material. What should the dimensions of the rectangular field be?



Solution

Let x be the width (in metres) of the field and y be the length (in metres) of the field.
Let L be the length of the fencing that is to be minimized.

$$A = xy$$
$$1800 = xy$$
$$y = \frac{1800}{x}$$

The total length of a fence is $L = 2x + y$. Substitute $y = \frac{1800}{x}$ to give a function in one variable.

$$L = 2x + \frac{1800}{x}$$

The derivative of the function, equated to zero, will give a critical value for x; then solve for y.

$$\frac{dL}{dx} = 2 - 1800x^{-2} = 0$$
$$2 = 1800x^{-2}$$
$$x^2 = 900$$
$$x = 30$$

x cannot be negative.

When $x = 30$, $y = \frac{1800}{30}$
$= 60$

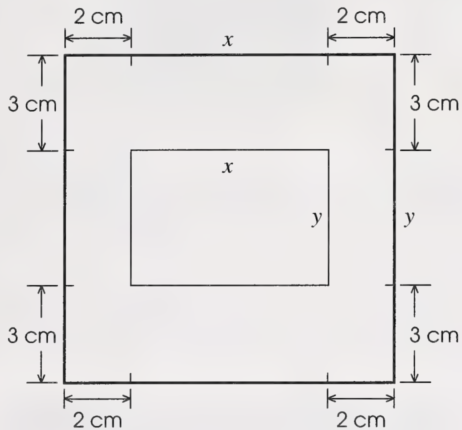
When $x = 30$, $\frac{d^2L}{dx^2} = 3600x^{-3}$
$= \frac{3600}{x^3}$
$= \frac{3600}{27\,000}$ (positive)

Therefore, L is a minimum at $x = 30$.

The dimensions of the rectangular field should be 30 m by 60 m.

Example 6

A rectangular page is to contain 150 cm^2 of printing. The margins at the top and bottom of the page are 3 cm. The margins at each side are 2 cm. What should the dimensions of the page be if the minimum amount of paper is used?



Solution

Let x be the width of the printed area, y be the length of the printed area, $x+4$ be the width of the page, and $y+6$ be the length of the page.

Let A be the area to be minimized.

The problem has two equations: $xy=150$ and $A=(x+4)(y+6)$.

Solving the first equation for y produces $y=\frac{150}{x}$. Substitute y into the second equation.

$$\begin{aligned} A &= (x+4)\left(\frac{150}{x}+6\right) \\ &= 150+6x+\frac{600}{x}+24 \\ &= 174+6x+\frac{600}{x} \end{aligned}$$

To find the minimum area, take the derivative of the function and equate it to zero.

$$\begin{aligned} \frac{dA}{dx} = 6-\frac{600}{x^2} &= 0 \\ 6 &= \frac{600}{x^2} \\ x^2 &= 100 \\ x &= 10 \end{aligned}$$

Since x is a measure, it cannot be negative.

$$\begin{aligned} \text{When } x=10,\ y &= \frac{150}{x} \\ &= 15 \end{aligned}$$

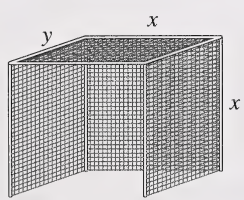
The second derivative, $\frac{d^2A}{dx^2}=\frac{1200}{x^3}$, is positive when $x=10$; thus, the area is a minimum.

The dimensions of the page should be 14 cm by 21 cm.

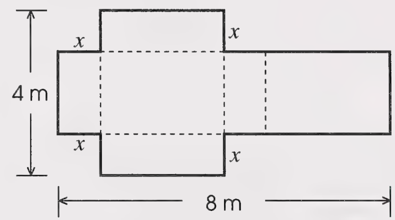
6. A farmer wants to build two adjacent rectangular cattle pens on 60 000 m^2 of land. The farmer also wants these cattle pens to be of equal area. What is the least amount of fence needed?

7. A page contains 600 cm^2. The margins at the top and bottom are 3 cm. The margins at each side are to be 2 cm. What are the dimensions of the paper if the printed area is a maximum?

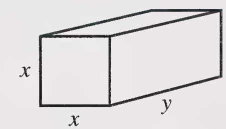
8. A net enclosure for batting practice is open at one end. What dimensions, requiring the least amount of netting, will enclose a volume of 144 m^3?



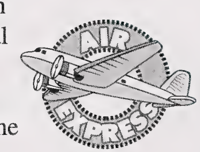
9. A rectangular piece of cardboard, 4 m by 8 m, is cut and folded on the dotted lines to form a closed box (as shown in the diagram). If x is the width of the flaps, write an expression, in terms of x, that represents the volume; then find the value of x that will maximize the volume.



10. Rectangular boxes shipped by a certain air courier service are subject to a size restriction. The sum of the three dimensions of the box cannot exceed 90 cm. If the box has square ends, what dimensions would maximize its volume?



11. A similar restriction, by the air courier, given in the previous question applies to cylindrical packages. Half the circumference plus the height of the cylinder cannot exceed 90 cm. What dimensions would maximize the volume of such a cylindrical package?



Check your answers by turning to the Appendix.

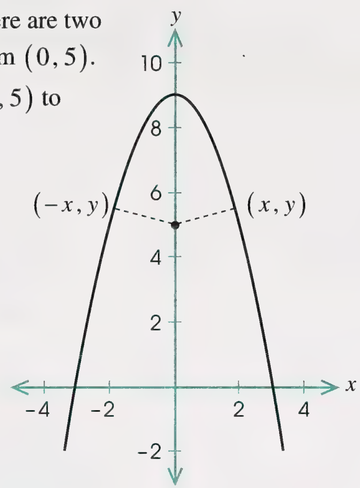
Example 7

Find the points on the graph of $y = 9 - x^2$ that are closest to the point $(0, 5)$.

Solution

From the sketch, it seems that there are two points at a minimum distance from $(0, 5)$. The distance d from the point $(0, 5)$ to the point (x, y) is

$d = \sqrt{(x-0)^2 + (y-5)^2}$.



If (x, y) lies on the parabola, then y can be replaced with $9 - x^2$.

$$d = \sqrt{x^2 + \left(9 - x^2 - 5\right)^2}$$
$$= \sqrt{x^2 + \left(4 - x^2\right)^2}$$

Instead of minimizing d, you can minimize the simpler expression d^2. (d is the smallest when d^2 is the smallest.) Therefore, you need to find the critical numbers for the function.

$$f(x) = x^2 + \left(4 - x^2\right)^2$$
$$= x^4 - 7x^2 + 16$$

$$f'(x) = 4x^3 - 14x = 0$$
$$2x\left(2x^2 - 7\right) = 0$$

$$2x = 0 \quad \text{or} \quad 2x^2 - 7 = 0$$
$$x = 0 \qquad\qquad x = \pm\sqrt{\frac{7}{2}}$$

At $x = 0$, $f''(x) = 12x^2 - 14$
$$= -14$$

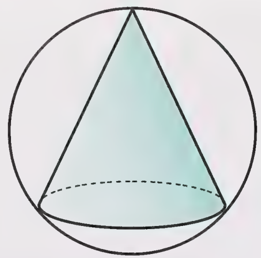
This denotes a maximum at $x = 0$. At $x = +\sqrt{\frac{7}{2}}$ and $x = -\sqrt{\frac{7}{2}}$ $f''(x)$ is positive. Therefore, $x = \pm\sqrt{\frac{7}{2}}$.

When $x = \pm\sqrt{\frac{7}{2}}$, $y = 9 - \frac{7}{2}$
$$= \frac{11}{2}$$

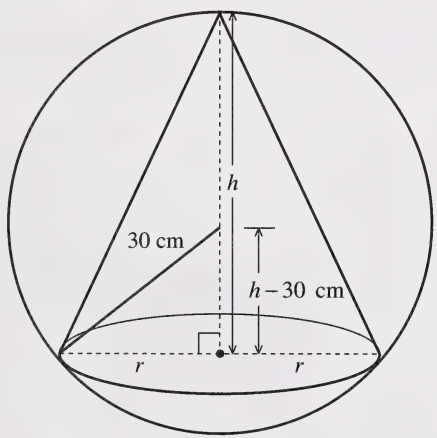
The points closest to $(0, 5)$ are $\left(\sqrt{\frac{7}{2}}, \frac{11}{2}\right)$ and $\left(-\sqrt{\frac{7}{2}}, \frac{11}{2}\right)$.

Example 8

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 30 cm.



Solution



Let h be the height of the right circular cone and r be the radius of the base of the cone.

Let V be the volume to be maximized.

You can use the Pythagorean Theorem to write r^2 in terms of h.

$$30^2 = r^2 + (h-30)^2$$
$$900 = r^2 + h^2 - 60h + 900$$
$$r^2 = 60h - h^2$$

Rewrite the volume formula in one variable.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(60h - h^2\right)h \\ &= \frac{1}{3}\pi\left(60h^2 - h^3\right) \end{aligned}$$

The derivative of the volume gives you a critical value for h.

$$\begin{aligned} \frac{dV}{dh} = \frac{1}{3}\pi\left(120h - 3h^2\right) &= 0 \\ 120h - 3h^2 &= 0 \\ h(120 - 3h) &= 0 \end{aligned}$$

$$h = 0 \text{ or } 120 - 3h = 0$$
$$3h = 120$$
$$h = 40$$

$$\frac{d^2V}{dh^2} = 40\pi - 2\pi h$$

At $h = 40$, $\frac{d^2V}{dh^2} = -40\pi$, which indicates a maximum.

When $h = 40$, $r^2 = 60h - h^2$
$$= 60(40) - 40^2$$
$$= 2400 - 1600$$
$$= 800$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi \times 800 \times 40$$
$$= \frac{32\,000}{3}\pi$$
$$\doteq 33\,510$$

The volume of the largest right circular cone is $33\,510 \text{ cm}^3$.

Watch the video titled *Maxima and Minima*, from the *Catch 31* series, ACCESS Network. This program will show you further applications and examples of maximum and minimum problems. This video is available from the Learning Resources Distributing Centre.

These questions require the concepts modelled by the examples; try them.

12. If a trapezoid has its longer base on the x-axis and the other two vertices inscribed in $y = 4 - x^2$, find the maximum area of the trapezoid.

13. If an isosceles triangle is inscribed in a circle of radius 4 cm, find the dimensions of the isosceles triangle of maximum area.

14. Find the maximum volume of the largest right circular cone that can be inscribed in a sphere of radius 12 cm.

15. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius 30 cm.

16. Find the dimensions of a rectangle of maximum area inscribed in a circle of radius 6 cm.

Check your answers by turning to the Appendix.

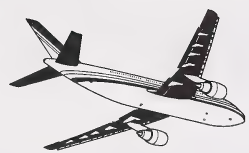
What other considerations besides maximum area would a farmer take into account, if there is a fixed amount of fencing to enclose a rectangular area of land?

Activity 3: Extreme Values of Distances and Times

Describing the positions and paths of moving objects is an application of calculus. To facilitate your work, the problems will be organized around a coordinate system. The position of an object can be described in relation to the origin.

Example 1

An airplane flying west at 200 km/h passes a house 10 min before another plane, flying south at 300 km/h, passes over the same point. Assuming that both planes are at the same altitude, at what time is the distance between them a minimum? What is that distance? Round your answer to the nearest hundredth.

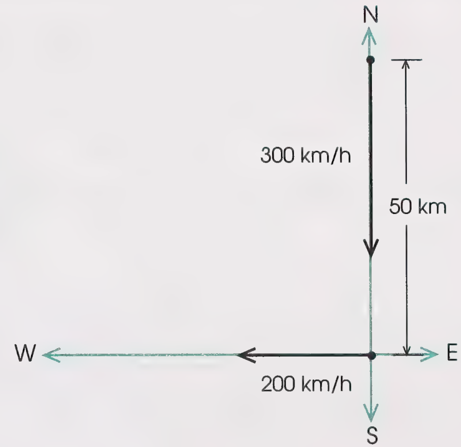


Solution

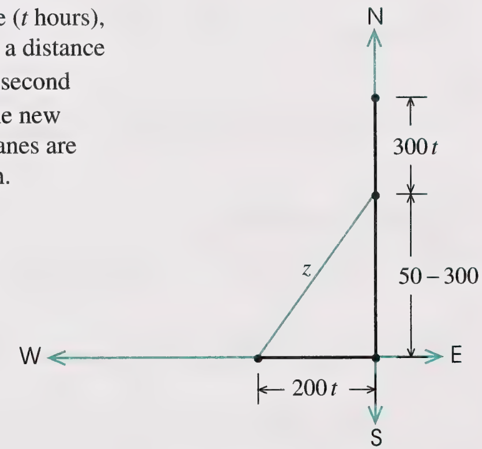
Let x be the position of the first plane and y be the position of the second plane.

Initially, the first plane is at the origin $(x = 0)$. The second plane, travelling at a speed of 300 km/h for 10 min, covers a distance of $y = 300\left(\frac{10}{60}\right) = 50$ km. Thus, the second plane is 50 km north of the house initially.

The initial situation can be shown as follows:



After a period of time (t hours), the first plane travels a distance of $200t$ km, and the second travels $300t$ km. The new positions of the airplanes are shown in the diagram.



The distance z between the two planes is given by $z^2 = x^2 + y^2$ where $x = 200t$ and $y = 50 - 300t$.

$$z^2 = (200t)^2 + (50 - 300t)^2$$

The distance between the two planes will be at a minimum when $\frac{dz}{dt} = 0$.

$$\begin{aligned}
\frac{dz}{dt} \bullet z^2 &= \frac{d}{dt}(200t)^2 + \frac{d}{dt}(50 - 300t)^2 \\
2z \bullet \frac{dz}{dt} &= 2(200t)(200) + 2(50 - 300t)(-300) \\
\frac{dz}{dt} &= \frac{40\,000t - 15\,000 + 90\,000t}{z} \\
0 &= \frac{130\,000t - 15\,000}{z} \\
130\,000t &= 15\,000 \\
t &= \frac{15\,000}{130\,000} \\
&\doteq 0.12 \text{ h or } 6.9 \text{ min}
\end{aligned}$$

About 0.12 h after the first plane passes over the house, the distance between the planes is at a minimum.

When $t \doteq 0.12$, $x \doteq 200(0.12)$
$\doteq 24$

When $t \doteq 0.12$, $y = 50 - 300(0.12)$
$= 14$

$$\begin{aligned}
z^2 &\doteq (24)^2 + (14)^2 \\
z^2 &\doteq 576 + 196 \\
z^2 &\doteq 772 \\
z &\doteq 27.78
\end{aligned}$$

The minimum distance between the planes is about 27.78 km.

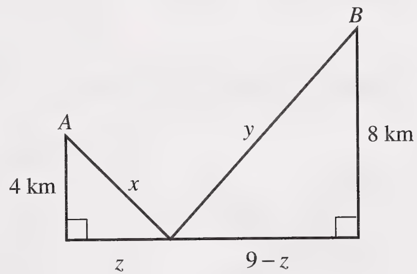
Example 2

Yacht A is anchored 4 km off a straight shoreline. Opposite a point 9 km down the coast, Yacht B is anchored 8 km off the shore. A boat from the Yacht A is to bring a passenger to the beach, then proceed to the Yacht B to pick up another passenger. At what point along the shore should the first passenger be dropped in order to minimize the distance?

Solution

Let x be the distance (in km) from Yacht A, y be the distance (in km) from Yacht B, and z be the distance (in km) along the shore.

Let D be the distance to be minimized.



The equation to be minimized is $D = x + y$.

Using the diagram and the Pythagorean Theorem, write x and y in terms of z.

$$x^2 = z^2 + 4^2$$
$$x = \sqrt{z^2 + 16}$$

$$y^2 = (9 - z)^2 + 8^2$$
$$y^2 = \left(z^2 - 18z + 81\right) + 64$$
$$y = \sqrt{z^2 - 18z + 145}$$

Now the minimizing equation can be written in terms of one variable.

$$\begin{aligned} D &= \sqrt{z^2+16}+\sqrt{z^2-18z+145} \\ &= \left(z^2+16\right)^{\frac{1}{2}}+\left(z^2-18z+145\right)^{\frac{1}{2}} \end{aligned}$$

Convert the square root to the power of $\frac{1}{2}$ to find the derivitive.

$$\frac{dD}{dz}=\frac{1}{2}\left(z^2+16\right)^{-\frac{1}{2}}(2z)+\frac{1}{2}\left(z^2-18z+145\right)^{-\frac{1}{2}}(2z-18)=0$$

$$\left(z^2+16\right)^{-\frac{1}{2}}(z)+\left(z^2-18z+145\right)^{-\frac{1}{2}}(z-9)=0$$

$$\frac{z}{\sqrt{z^2+16}}=\frac{-(z-9)}{\sqrt{z^2-18z+145}}$$

$$\frac{z}{\sqrt{z^2+16}}=\frac{(9-z)}{\sqrt{z^2-18z+145}}$$

Cross multiply and square both sides to remove the radicals.

$$\begin{aligned} z^2\left(z^2-18z+145\right) &= (9-z)^2\left(z^2+16\right) \\ z^4-18z^3+145z^2 &= \left(81-18z+z^2\right)\left(z^2+16\right) \\ z^4-18z^3+145z^2 &= 81z^2+1296-18z^3-288z+z^4+16z^2 \\ 48z^2+288z-1296 &= 0 \\ z^2+6z-27 &= 0 \\ (z+9)(z-3) &= 0 \end{aligned}$$

Reduce by 48.

$z+9=0$ or $z-3=0$

$z=-9$ $\quad z=3$

As a distance, $z=-9$ is invalid.

The boat should land 3 km from the point directly across from Yacht *A*.

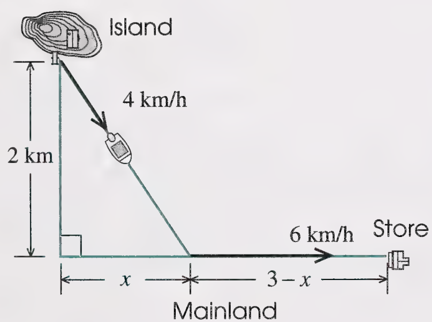
Example 3

A cabin is situated on an island 2 km from the nearest point on the mainland. A store on the mainland, where supplies are bought, is 3 km down the shoreline. If a motor boat travels at a speed of 4 km/h and the average person can walk 6 km/h, toward what point on the mainland should the boat be aimed in order to reach the store in the least time?

Solution

Let x be the distance along the shoreline to the point where the boat is headed.

Let T be the total time to be minimized.



The total time it takes to reach the store is found as follows:

$$T = \text{boating time} + \text{walking time}$$
$$= \frac{\text{boating distance}}{\text{speed}} + \frac{\text{walking distance}}{\text{speed}}$$

$t = \dfrac{d}{s}$

From the diagram, boating distance $= \sqrt{x^2 + 4}$ and walking distance $= 3 - x$.

Now you can write the total time in terms of one variable.

$$T = \frac{\sqrt{x^2+4}}{4} + \left(\frac{3-x}{6}\right)$$
$$= \frac{1}{4}\left(x^2+4\right)^{\frac{1}{2}} + \left(\frac{1}{2} - \frac{1}{6}x\right)$$

The derivative will identify the critical value for x.

$$\frac{dT}{dx} = \frac{1}{4} \bullet \frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}(2x) + \left(-\frac{1}{6}\right) = 0$$

$$\frac{x}{4\sqrt{x^2+4}} + \left(-\frac{1}{6}\right) = 0$$

Cross multiply and square.

$$\frac{1}{6} = \frac{x}{4\sqrt{x^2+4}}$$

$$16\left(x^2+4\right) = (6x)^2$$

$$16x^2 + 64 = 36x^2$$

$$20x^2 = 64$$

$$x^2 = \frac{64}{20}$$

$$x \doteq 1.8$$

Therefore, to minimize the time, the boat should be aimed approximately 1.8 km down the shore from a point directly across from the island; then, the remaining 1.2 km should be walked.

You have studied extreme values in distances and times; it's time now to try some questions on your own.

1. At 10:00 A.M., a cargo ship is sailing west at 20 knots. A customs boat 60 km due north of the ship is travelling south at 30 knots. At what time will the boats be closest to each other?

2. A biathalon is organized for swimmers and speed walkers. The first leg will require swimming from a launch 4 km from the shore of the lake. The second leg involves a walk to the finish line which is 8 km from a point on the shoreline directly across from the launch. To minimize the length of time of the race, to what point on the shore should a particular contestant aim if he or she swims at 3 km/h and walks at 5 km/h?

3. Given the race in the question 2, what is the distance of the race corresponding to the minimum time?

Note: Be careful! The minimum time as outlined in question 2 does not necessarily represent the minimum distance possible!

Check your answers by turning to the Appendix.

Activity 4: Extreme-Value Problems in Economics and Other Sciences

One of the primary objectives in any business venture is to maximize profits or revenue and to minimize costs. On the other hand, an engineer may want to maximize efficiency, or perhaps determine the maximum height of a projectile. In this section you will create functions for these quantities and use the derivative to determine their extreme values.

Economics

Remember that, when selling a product, total revenue or income depends on the selling price of the product and the number of units sold. The price of a product may be determined by the number of units sold. If a product is selling well, the price per unit may decrease.

If it costs a company $C(x)$ dollars to produce x units of a product, then the function defined by C is called the **cost function**. The total cost of producing x units of a product includes fixed costs (rent, utilities, depreciation on machinery, etc.), which continue even if nothing is produced, and variable costs (labour, materials, etc.), which depend on the number of items produced.

$$\text{Cost function} = \text{fixed costs} + \text{variable costs}$$

$$C(x) = F + xp(x)$$

In the cost funtion, x is the number of units produced and $p(x)$ is the price per unit.

Economists call the derivative of the cost function the **marginal cost**. This derivative is used to calculate minimum costs.

The **average cost function**, $c(x) = \frac{C(x)}{x}$, calculates the cost per unit when x units are produced. When the average cost is at a minimum, it equals marginal cost.

Example 1

A manufacturing company determines that the variable cost of producing a certain number of microwave ovens is described by the function $x(0.5x+120)$. If the fixed costs are calculated to be $20 000, how many microwave ovens should be produced in order for the total costs to be minimized?

Solution

Let x be the number of microwave ovens to be manufactured.
Let c be the average cost to be minimized.

$$\text{Total Cost} = \text{fixed costs} + \text{variable costs}$$

$$\begin{aligned} C(x) &= 20\,000 + x(0.5x+120) \\ &= 20\,000 + 0.5x^2 + 120x \end{aligned}$$

Determine the average cost function.

$$\begin{aligned} c(x) &= \frac{C(x)}{x} \\ &= \frac{20\,000 + 0.5x^2 + 120x}{x} \\ &= \frac{20\,000}{x} + 0.5x + 120 \end{aligned}$$

The minimum cost can be found in two ways:

Method 1

Find x when the derivative of the average cost is 0.

$$\begin{aligned}\frac{dc}{dx} = \frac{-20\,000}{x^2} + 0.5 &= 0\\ \frac{20\,000}{x^2} &= 0.5\\ x^2 &= \frac{20\,000}{0.5}\\ x^2 &= 40\,000\\ x &= 200\end{aligned}$$

$$\frac{d^2c}{dx^2} = \frac{40\,000}{x^3}$$

When $x = 200$, $\frac{d^2c}{dx^2}$ is positive; therefore, cost is a minimum.

Method 2

When marginal cost = average cost, the average cost will be a minimum.

$$C(x) = 20\,000 + 0.5x^2 + 120x$$

$$\frac{dC}{dx} = x + 120$$

The derivative of the cost function is marginal cost.

$$\begin{aligned}\frac{dC}{dx} &= c(x)\\ x + 120 &= \frac{20\,000}{x} + 0.5x + 120\\ 0.5x &= \frac{20\,000}{x}\\ x^2 &= 40\,000\\ x &= 200\end{aligned}$$

The cost is a minimum if 200 microwaves are manufactured.

Example 2

The cost of fuel used by a particular boat varies directly as the cube of the speed of the boat. It costs \$2.50/h when the speed is 5 km/h. If you want to rent this boat, it costs \$40/h plus fuel cost. What is the most economical speed for a voyage of 90 km? What is the minimum cost?

Solution

Let v be the speed of the boat and k be the constant of variation. Let C be the total cost to be maximized.

$$\begin{aligned}\text{Fuel cost per hour} &= kv^3\\ 2.5 &= k5^3\\ k &= \frac{2.5}{125}\\ &= 0.02\end{aligned}$$

direct variation equation as stated in the problem

The fuel cost per hour is $0.02v^3$.

Total Cost = (rent per hour + fuel cost per hour) × time

$$C = \left(40 + 0.02v^3\right)\left(\frac{90}{v}\right)$$
$$= \frac{3600}{v} + 1.8v^2$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

At a minimum,

$$\frac{dC}{dv} = -3600v^{-2} + 3.6v = 0$$
$$3.6v = 3600v^{-2}$$
$$3.6v^3 = 3600$$
$$v^3 = 1000$$
$$v = 10$$

Therefore, the most economical speed is 10 km/h.

When $v = 10$, $C = \frac{3600}{10} + 1.8(10)^2$
$$= 360 + 180$$
$$= 540$$

$$\frac{d^2C}{dv^2} = 7200v^{-3} + 3.6$$

At $v = 10$, $\frac{d^2C}{dv^2}$ is positive; therefore, it is a minimum.

The minimum cost is $540.

A business would not only be interested in minimizing costs, it would also like to maximize profits. A **revenue function**, the number of units sold multiplied by the price per unit, determines total revenue. The derivative of the revenue function is called **marginal revenue** and is used to calculate maximum profits.

Revenue = price per unit × number of units or $R(x) = p(x) \times x$

The profits realized are found by subtracting costs from revenue.

Profit = Revenue − Costs or $P(x) = R(x) - C(x)$

For maximum profit, marginal revenue equals marginal cost; that is, $R'(x) = C'(x)$.

Example 3

A tour company offers ski excursions at $160 per person for groups up to 50. For groups larger than 50, the company will reduce the cost of every ticket by $2 for each person in excess of 50. What size of group would produce the greatest revenue?

Solution

The total revenue depends on the price of the ticket and the number of people. If 50 people go skiing, then
Revenue = $\$160 \times 50 = \$8000$.

Let x be the number of people in excess of 50, $(50+x)$ be the number of people on the ski trip, and $(160-2x)$ be the reduced ticket price.

Let R be the revenue to be maximized.

Therefore, the revenue function is $R(x)=(160-2x)(50+x)$.

The maximum revenue is found when $\frac{dR}{dx}=0$.

product rule

$$\begin{aligned}\frac{dR}{dx}=(160-2x)(1)+(50+x)(-2)&=0\\160-2x-100-2x&=0\\4x&=60\\x&=15\end{aligned}$$

$\frac{d^2R}{dx^2}=-4$; therefore, $x=15$ yields a maximum.

The company would earn the greatest revenue with a group of $50+15=65$ skiers. The tickets would cost $160-2x(15)=\$130$ per person, and the company's total revenue would be \$8450.

Example 4

A toy retailer finds that 100 teddy bears can be sold if each is priced at \$12. However, twenty more can be sold for every \$1 decrease in price. The cost of each bear is \$6. In addition, the retailer must pay a fixed cost of \$150. How should the bears be priced in order to realize the most profit?

Solution

Let x be the selling price for each bear.
Let P be the profit to be maximized.

$$\begin{aligned}\text{Number of bears sold}&=100+20(12-x)\\&=340-20x\end{aligned}$$

$\therefore$ Total Sales $=x(340-20x)$

$$\begin{aligned}\text{Total Cost}&=150+(340-20x)(6)\\&=2190-120x\end{aligned}$$

There are two methods to find the critical value:

Method 1: Using the Derivative of the Profit Formula

Profit = Total Sales – Total Cost

$$\begin{aligned}P&=x(340-20x)-(2190-120x)\\&=340x-20x^2-2190+120x\\&=-20x^2+460x-2190\end{aligned}$$

$$\begin{aligned}\frac{dP}{dx}=-40x+460&=0\\40x&=460\\x&=11.5\end{aligned}$$

$\frac{d^2P}{dx^2}=-40$; therefore, $x=11.5$ yields a maximum.

Method 2: Using Marginal Revenue and Marginal Cost

$$\boxed{\frac{dR}{dx} = \frac{dC}{dx}}$$

Revenue = number of items × price per item

$$\begin{aligned} R(x) &= x[p(x)] \\ &= x(340 - 20x) \\ &= 340x - 20x^2 \end{aligned}$$

$$\frac{dR}{dx} = 340 - 40x$$

Total Cost = fixed costs + variable costs

$$\begin{aligned} C(x) &= 150 + (340 - 20x)(6) \\ &= 2190 - 120x \end{aligned}$$

$$\frac{dC}{dx} = -120$$

Marginal Revenue = Marginal Cost

$$\begin{aligned} 340 - 40x &= -120 \\ 40x &= 460 \\ x &= 11.5 \end{aligned}$$

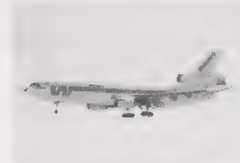
When $x = 11.5$, $P = -20(11.5)^2 + 460(11.5) - 2190$

$$= 455$$

The bears should be priced at $11.50 each to realize a maximum profit of $455.

The following questions give you a chance to practice further economic and scientific applications.

1. The cruising speed of an airplane is 300 km/h. The formula $C = 250 + \frac{h}{10} + \frac{2\,500\,000}{h}$ represents the cost per hour of flying this airplane, where h is the height in metres. Determine the height at which the cost of flying is a minimum.



2. An apartment management firm tends an apartment block containing 150 units. All 150 units are rented at $460 per unit with each unit costing $72.50/month for utility and repairs. For every $25 rent increase, four fewer apartments are occupied. What rent should be charged in order to realize the most profit? (Assume that rent would only be increased by a multiple of $25.)

3. A motel has 40 rooms that cost $60 per night. For each $5 per night increase, they have 1 additional room left vacant. What should the nightly rate be to maximize the motel's total revenue? What is the maximum revenue?

4. During the summer, Dana offers a lawn-mowing service. Last summer Dana charged $15 per job and averaged 15 customers per week. If she increased her price by $2, she lost 1 customer per week. If her overhead per job is $5, what should she charge in order to maximize her profits? What is her maximum profit?

Check your answers by turning to the Appendix.

Other Sciences

Engineers are concerned that their equipment works efficiently and that their products have a maximum strength and use. Biologists study maximums and minimums concerning population growth. Look at the following examples to see some applications of these types of extremes.

Example 1

The current in a battery is $I = \frac{E}{R+r}$, where E is the electromotive force, R is the external resistance, and r is the internal resistance. E and r are characteristics of the battery and, therefore, are constants. The power developed is $P = RI^2$. If the internal resistance of a battery is $0.15\ \Omega$, what is R when P is a maximum?

Solution

Ω = ohm, a unit of resistance

$$\begin{aligned} P &= RI^2 \\ &= R\left(\frac{E}{R+r}\right)^2 \\ &= \frac{RE^2}{R^2 + 2rR + r^2} \end{aligned}$$

If $r = 0.15\ \Omega$, then $P = \dfrac{E^2 R}{R^2 + 0.3R + 0.0225}$.

$$\frac{dP}{dR} = \frac{\left(R^2 + 0.3R + 0.0225\right)D_R\left(E^2R\right) - \left(E^2R\right)D_R\left(R^2 + 0.3R + 0.0225\right)}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

$$= \frac{\left(R^2 + 0.3R + 0.0225\right)\left(E^2\right) - \left(E^2R\right)(2R + 0.3)}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

$$= \frac{E^2R^2 + 0.3RE^2 + 0.0225E^2 - 2E^2R^2 - 0.3E^2R}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

$$= \frac{-E^2R^2 + 0.0225E^2}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

To find the maximum, let $\frac{dP}{dR} = 0$.

$$\begin{aligned} -E^2R^2 + 0.0225E^2 &= 0 \\ E^2R^2 &= 0.0225E^2 \\ R^2 &= 0.0225 \\ R &= 0.15 \end{aligned}$$

R cannot be negative.

$$\frac{dP}{dR} = \frac{-E^2R^2 + 0.0225E^2}{\left(R^2 + 0.3R + 0.0225\right)^2}$$

$$= \left(-E^2R^2 + 0.0225E^2\right)\left(R^2 + 0.3R + 0.0225\right)^{-2}$$

$$\frac{d^2P}{dR^2} = \left(-E^2R^2 + 0.0225E^2\right) D_R \left(R^2 + 0.3R + 0.0225\right)^{-2} + \left(R^2 + 0.3R + 0.0225\right)^{-2} D_R \left(-E^2R^2 + 0.0225E^2\right)$$

$$= \left(-E^2R^2 + 0.0225E^2\right)(-2)\left(R^2 + 0.3R + 0.0225\right)^{-3}(2R + 0.3) + \left(R^2 + 0.3R + 0.0225\right)^{-2}\left(-2E^2R\right)$$

If $R = 0.15$, then $\frac{d^2P}{dR^2} = 0 - \frac{0.3E^2}{(0.09)^2}$

$\doteq -37.037E^2 \quad (\text{negative})$

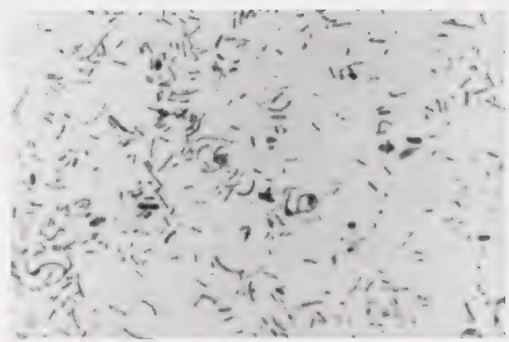
Therefore, when $R = r = 0.15\,\Omega$, P is a maximum.

Example 2

When a population of N bacteria is introduced into a nutrient medium, the population grows according to the equation $P = N + \frac{Nt}{81+t^2}$, where t is the number of hours of growth. If $N = 5000$, determine the maximum size of this population and the time in which that size is realized.

Solution

When $N = 5000$, $P = 5000 + \frac{5000t}{81+t^2}$.



$$\frac{dP}{dt} = \frac{\left(81+t^2\right)D_t\left(5000t\right)-\left(5000t\right)D_t\left(81+t^2\right)}{\left(81+t^2\right)^2}$$

$$= \frac{\left(81+t^2\right)(5000)-(5000t)(2t)}{\left(81+t^2\right)^2}$$

$$= \frac{405\,000+5000t^2-10\,000t^2}{\left(81+t^2\right)^2}$$

$$= \frac{405\,000-5000t^2}{\left(81+t^2\right)^2}$$

For the quadratic expression $405\,000-5000t^2$, $\frac{dP}{dt}>0$ when $t<9$, and $\frac{dP}{dt}<0$ when $t>9$. Therefore, a maximum occurs.

At a maximum, $\frac{dP}{dt}=0$.

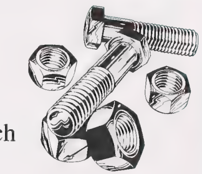
$$\begin{aligned} 405\,000-5000t^2 &= 0 \\ 5000t^2 &= 405\,000 \\ t^2 &= 81 \\ t &= 9 \end{aligned}$$

t cannot be negative.

$$\text{When } t=9,\ P = 5000+\frac{5000(9)}{81+81}$$
$$= 5000+\frac{45\,000}{162}$$
$$\doteq 5278$$

The maximum size of this population is approximately 5278, and this occurs when $t=9$ h.

Now try these questions.



5. The efficiency of a bolt is represented by $E=\frac{\left(m-\frac{1}{3}m^2\right)}{\left(m+\frac{1}{3}\right)}$, where m is the tangent of the pitch angle of the bolt. Determine m for which E is a maximum.

6. The current in a battery is $I=\frac{E}{R+r}$, where E is the electromotive force, R is the external resistance, and r is the internal resistance. E and r are constants. The power developed is $P=I^2R$. What is R when $r=0.2\ \Omega$ and the power is a maximum?

7. When a population of N bacteria is introduced into a nutrient medium, the population grows according to the equation $P=N+\frac{Nt}{t^2+64}$, where t is the number of hours of growth. If $N=2000$, determine the maximum size of this population and when that maximum population occurs.

Check your answers by turning to the Appendix.

As you can see, extreme-value problems occur not only in the physical sciences but in the biological sciences as well!

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In solving problems that involve extreme values, often the greatest challenge is to change the word problem to a maximum or minimum problem. You must set up a function that is to be maximized or minimized. Follow these steps:

Step 1: Read the problem carefully—more than once is often necessary. From the description of the situation, identify the unknown(s), the given quantities, and the given condition.

Step 2: If it is feasible, draw a diagram and label it with given and required quantities.

Step 3: Select symbols or variables to represent the unknowns and the quantity to be maximized or minimized. It is helpful to use symbols that suggest the unknown, like d for distance, ℓ for length, t for time, and so on. Write statements that identify the variables with their unknowns.

Step 4: Write an equation expressing the quantity to be minimized or maximized. This equation will often be in terms of two variables. The problem will give some condition that names a relationship between the two variables. Use this condition to express one variable in terms of the other; then rewrite the function to be maximized or minimized in terms of one varaible.

Step 5: Take the derivative of the function, set it equal to zero, and solve to find the maximum or minimum value(s). Determine if the solution(s) is part of the domain of the function.

Step 6: Use the result(s) from Step 5 to deterimine the values of the other unknowns by substitution.

Step 7: Verify that your result is a possible maximum or minimum value using an appropriate test (such as the first or second derivative tests).

Step 8: Write a concluding statement, ensuring that you have answered the question(s) posed in the problem.

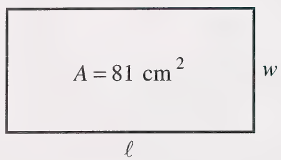
Example

The area of a rectangle is $81\ \text{cm}^2$. Find the dimensions of the rectangle if its perimeter is a minimum. What is the minimum perimeter?

Solution

Step 1: Read the problem carefully.

Step 2: Draw a diagram.



Step 3: Let ℓ be the length of the rectangle and w be the width of the rectangle.

Let P be the perimeter to be minimized.

Step 4:

$$A = \ell \times w \qquad P = 2\ell + 2w$$

$$81 = \ell \times w \qquad = 2\ell + 2\left(\frac{81}{\ell}\right)$$

$$w = \frac{81}{\ell} \qquad = 2\ell + 162\ell^{-1}$$

Step 5: $\frac{dP}{d\ell} = 2 + (-1)(162)\ell^{-2}$

$$= 2 - 162\ell^{-2}$$

At a minimum, $\frac{dP}{d\ell} = 0$.

$$2 - 162\ell^{-2} = 0$$
$$2 = 162\ell^{-2}$$
$$\ell^2 = 81$$
$$\ell = 9$$

ℓ is always positive; so -9 is not a part of the domain.

Step 6: Substitute ℓ into $w = \frac{81}{\ell}$. Thus, $w = \frac{81}{\ell} = 9$ cm.

Substitute ℓ and w into $P = 2(\ell + w)$.

$$P = 2(9+9)$$
$$= 36$$

Step 7: $\frac{d^2P}{d\ell^2} = 324\ell^{-3}$

At $\ell = 9$, $\frac{d^2P}{d\ell^2} = \frac{324}{729}$ (positive)

Therefore, $P = 36$ cm is a minimum.

Step 8: The dimensions of the rectangle that make its perimeter a minimum are 9 cm by 9 cm (a square). The minimum perimeter is 36 cm.

1. The perimeter of a rectangle is 100 cm. Find the dimensions of the rectangle if its area is a maximum. What is the maximum area?

2. An open pencil box is to be constructed with a square base. The total surface area of all four sides and the base is to be 48 cm^2. Determine the dimensions of the box if its volume is a maximum. What is the maximum volume?

Check your answers by turning to the Appendix.

Enrichment

To determine the maximum or minimum value of a function, you have to find the zeros of its first derivative. Sometimes it is not that easy. If the function is of a degree higher than 3, you may find it very difficult to factor its first derivative. Look at the following example.

Example

Find the maximum and/or minimum values of the function

$y = 3x^4 - 1x^3 - 6x^2 + 3x + 5.$

Solution

$$\frac{dy}{dx} = 12x^3 - 3x^2 - 12x + 3 = 0$$
$$4x^3 - x^2 - 4x + 1 = 0$$
$$x^2(4x-1) - 1(4x-1) = 0$$
$$\left(x^2 - 1\right)(4x-1) = 0$$
$$(x-1)(x+1)(4x-1) = 0$$

Factor by grouping or use synthetic division.

$x - 1 = 0$ or $x + 1 = 0$ or $4x - 1 = 0$

$x = 1$ $\quad x = -1$ $\quad x = \frac{1}{4}$

$$\frac{d^2y}{dx^2} = 36x^2 - 6x - 12$$

When $x = 1$, $y = 3(1)^4 - (1)^3 - 6(1)^2 + 3(1) + 5$
$= 3 - 1 - 6 + 3 + 5$
$= 4$

$$\frac{d^2y}{dx^2} = 36(1)^2 - 6(1) - 12$$
$$= 18 \quad (\text{positive})$$

Thus, 4 is a minimum value.

When $x = -1$, $y = 3(-1)^4 - (-1)^3 - 6(-1)^2 + 3(-1) + 5$
$= 3 + 1 - 6 - 3 + 5$
$= 0$

$$\frac{d^2y}{dx^2} = 36(-1)^2 - 6(-1) - 12$$
$$= 36 + 6 - 12$$
$$= 30 \quad (\text{positive})$$

Thus, 0 is a minimum value.

When $x=\frac{1}{4}$, $y=3\left(\frac{1}{4}\right)^4-\left(\frac{1}{4}\right)^3-6\left(\frac{1}{4}\right)^2+3\left(\frac{1}{4}\right)+5$

$$=\frac{3}{256}-\frac{1}{64}-\frac{6}{16}+\frac{3}{4}+5$$

$$=\frac{3-4-96+192+1280}{256}$$

$$=\frac{1375}{256}$$

$$=5\frac{95}{256}$$

$$\frac{d^2y}{dx^2}=36\left(\frac{1}{4}\right)^2-6\left(\frac{1}{4}\right)-12$$

$$=\frac{36}{16}-\frac{6}{4}-12$$

$$=-11\frac{1}{4} \quad (\text{negative})$$

Thus, $5\frac{95}{256}$ is a maximum value.

The function has minimum values at $x=1$ and $x=-1$, and a maximum value at $x=\frac{1}{4}$.

Try the following.

Find the maximum and/or minimum values of the following functions.

1. $y=x^4+4x^3-8x^2-48x+4$

2. $y=\frac{x^4}{2}+x^3-4x^2+3x-1$

Check your answers by turning to the Appendix.

Conclusion

In this section you have studied maximum and minimum problems as an application of the derivative. You used the derivative to determine extreme values in situations that involved pairs of numbers, geometric figures, distances, times, and the sciences. By working through the examples and doing the questions, you are now more aware of the many practical applications of calculus to business and science.

The honeybee solves an extreme-value problem when it constructs a honeycomb. The hexagonal chambers maximize the space for storage and expend the least amount of energy in doing so. Make a beeline to the next section, and discover another use for the derivative.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 2: Rates of Change

ALBERTA GOVERNMENT PUBLIC AFFAIRS BUREAU

Diving is very precise sport. Perfect or near perfect timing makes the difference between a good dive and an excellent dive. The swan dive describes a graceful curve through the air. The path of the swan dive follows an arc that initially travels upward, maximizes, and then descends to the water.

Divers are concerned with the mathematics of the dive. When will they hit the water? How fast will they be travelling when they hit? These are functions of time. There are many instances where distances and times vary. A runner does not maintain a constant rate throughout a long-distance race. Swimmers will control their speed in order to conserve energy for a burst of speed at the end of their race. In the field of merchandizing, a change in price will affect sales. These example show the importance of knowing that a change in one variable may have an affect on another.

In Module 3 you discovered that the derivative determines the slopes of tangents. In the previous module, you used the derivative as a tool for graphing relations. In Section 1 of this module you used the derivative to solve maximum and minimum problems. In this section you will look at the derivative as the measure of the relationship between changes in variables. In this application the derivative is a rate of change.

Activity 1: Velocity

Suppose an object is moving in a straight line. You might think of this as a race car speeding along a straightaway of a racetrack or a ball being thrown vertically upward or a brick being dropped from the top of a building. As time changes, distance changes.

In describing the motion of an object, a horizontal or vertical line with a designated origin is used to represent the pattern of motion. For instance, if you move an object along a straight line a distance of 6 cm and then move it backward 2 cm, the total distance travelled is $6 \text{ cm} + 2 \text{ cm} = 8 \text{ cm}$. The object is now 4 cm from its starting point. This 4 cm is called **displacement**. When travelling from one position to another, an object may move directly there or it may travel in a variety of directions before ending up in the second position. Therefore, the distance travelled may not be the same as the displacement. Movement to the right or upward is considered to be in the positive direction; movement to the left or downward is in the negative direction.

Distance travelled per unit of time is called **speed**, and displacement travelled per unit of time is called **velocity**.

$$\text{speed} = \frac{\text{distance}}{\text{time}} \qquad \text{velocity} = \frac{\text{displacement}}{\text{time}}$$

Distance and displacement are equivalent when an object travels in a straight line without any change in direction. Therefore, speed and velocity are equivalent under the same conditions.

For instance, if the preceding object can be moved in 2 s, then

$$\begin{aligned}\text{average speed} &= \frac{8 \text{ cm}}{2 \text{ s}} \\ &= 4 \text{ cm/s}\end{aligned} \qquad \begin{aligned}\text{average velocity} &= \frac{4 \text{ cm}}{2 \text{ s}} \\ &= 2 \text{ cm/s}\end{aligned}$$

The term *average* is used here because in practice, speed and velocity are seldom truly uniform for more than a very short period of time. You will study this concept further.

The function s that defines the position, relative to the origin, of an object as a function of time t is called a **position function**. If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t+\Delta t) - s(t)$ (recall first principles in Module 3), then by the ratio, given previously, for velocity

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

The average rate of change is called **average velocity**. A secant line could be used to represent the situation connecting the points $(t_1, s(t_1))$ and $(t_2, s(t_2))$. Thus, the slope of a secant is equal to average velocity.

As the distance between the positions $s(t_2)$ and $s(t_1)$ approaches zero, the secant becomes the slope of a tangent at a point. (Refer to Module 3.) The average velocity becomes the velocity at a given time. This velocity is called **instantaneous velocity**.

Since $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ is the derivative of s with respect to t, then $v(t)$ is defined as $v(t) = \frac{ds}{dt}$ at time t.

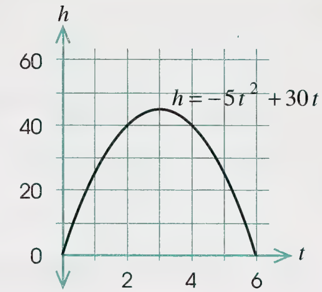
Therefore, the velocity of a moving object at any instant is the first derivative with respect to time of the displacement function. A positive velocity indicates motion to the right or upward, and a negative velocity indicates motion to the left or downward relative to a fixed point.

Example 1

If you throw a rock vertically upward from ground level with an initial velocity of 30 m/s, it will stop travelling upward at some point and fall back toward the ground. The height h (in metres) of the rock is a function of time t (in seconds) defined by

$h = -5t^2 + 30t$, where $t \geq 0$.

The following is a graph of the function.



- Determine the initial position of the rock.
- Determine the displacement from time $t = 1$ s to $t = 4$ s.
- What is the average velocity from $t = 1$ s to $t = 4$ s?
- What is the instantaneous velocity at time $t = 4$ s?
- When does the rock reach its maximum height? What is the rock's maximum height?

Solution

The initial position of the rock occurs when $t = 0$.

$$\begin{aligned} h(0) &= -5(0)^2 + 30(0) \\ &= 0 \end{aligned}$$

The initial position of the rock is 0 m.

$$\begin{aligned}\text{displacement} &= h(4)-h(1)\\ &= \left[-5(4)^2+30(4)\right]-\left[-5(1)^2+30(1)\right]\\ &= 40-25\\ &= 15\end{aligned}$$

The displacement is 15 m.

$$\begin{aligned}v_{ave} &= \frac{\text{displacement}}{\text{change in time}}\\ &= \frac{15}{3}\\ &= 5\end{aligned}$$

The average velocity is 5 m/s.

$$\begin{aligned}v(t) &= \frac{dh}{dt}\\ &= -10t+30\end{aligned}$$

When $t=4$, $v(t) = -10(4)+30$
$= -10$ m/s

The rock is falling at a velocity of 10 m/s at $t=4$ s.

The rock reaches its maximum height at $v(t)=0$.

$$\begin{aligned}v(t) = -10t+30 &= 0\\ 10t &= 30\\ t &= 3\end{aligned}$$

Recall from Section 1 that extreme values occur when the first derivative is zero.

When $t=3$, $h = -5(3)^2+30(3)$
$= -45+90$
$= 45$

The rock reaches a maximum height of 45 m when $t=3$ s.

Example 2

A particle begins moving along a number line from a point *P*, 10 units to the right of the origin. The displacement *s* from the origin is given by $s=2t^2-8t+10$, where $t \geq 0$ s.

- Find the average velocity between $t=3$ and $t=5$.
- Find the velocity at any time *t*.
- Find the velocity at $t=6$.
- Find the minimum value of *s*.
- Sketch the graph of $s=2t^2-8t+10$, and describe the motion of the particle.
- Find the total distance travelled in the first four seconds.

Solution

When $t = 3,\ s = 2(3)^2 - 8(3) + 10$
$= 4$

When $t = 5,\ s = 2(5)^2 - 8(5) + 10$
$= 20$

$$\begin{aligned} v_{ave} &= \frac{\Delta s}{\Delta t} \\ &= \frac{20-4}{5-3} \\ &= \frac{16}{2} \\ &= 8 \end{aligned}$$

The average velocity between $t = 3$ and $t = 5$ is 8 units/s.

The velocity at any time is equal to the first derivative of the function.

$$\frac{ds}{dt} = 4t - 8, \text{ where } t \geq 0$$

When $t = 6,\ \frac{ds}{dt} = 4(6) - 8$
$= 16$

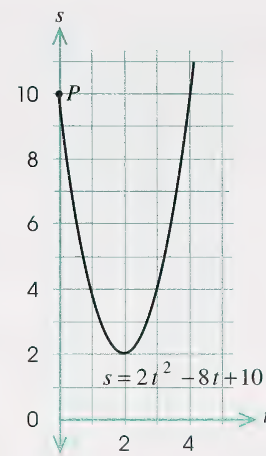
The velocity at $t = 6$ is 16 units/s.

$$\begin{aligned} \text{Let } 4t - 8 &= 0 \\ 4t &= 8 \\ t &= 2 \end{aligned}$$

When $t = 2,\ s = 2(2)^2 - 8(2) + 10$
$= 8 - 16 + 10$
$= 2$

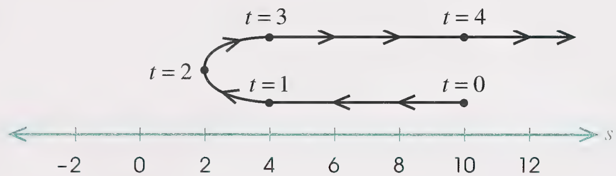
Therefore, the minimum value of s is 2 units.

t	s
0	10
1	4
2	2
3	4
4	10



The particle moves to the left from its initial position of 10 units until it reaches 2 units. It then reverses its direction and goes to the right. You can see that the particle starts out at a point 10 units from the origin and begins to move toward the origin. When the displacement between the particle and the origin is 2 units, the particle reverses its direction and moves away from the origin.

A schematic drawing representing the motion of the particle with respect to the number line is another way of tracking the movement of the particle.



The turning point at $t = 2$ occurs when $v = 0$.

Since the particle changes direction at $t = 2$ s, the distance must be divided into two parts.

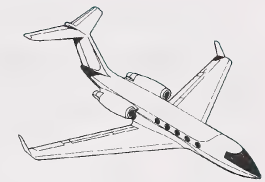
$$|s(2) - s(0)| = |2 - 10| = 8$$
$$|s(4) - s(2)| = |10 - 2| = 8$$

Thus, the total distance travelled in 4 s is $8 + 8 = 16$ units. This can also be read from the previous diagram.

Example 3

An airplane flies 1200 km from City A to City B at 600 km/h and returns at 400 km/h. What is the average velocity for the round trip? What is the average speed for the round trip?

Solution



Displacement = 0

$$\begin{aligned}\text{Total Time} &= \frac{1200}{600} + \frac{1200}{400} \\ &= 2 + 3 \\ &= 5 \text{ h}\end{aligned}$$

$$\begin{aligned}v_{ave} &= \frac{\text{displacement}}{\text{time}} \\ &= \frac{0}{5} \\ &= 0\end{aligned}$$

The average velocity is 0 km/h.

$$\begin{aligned}\text{Total Distance} &= 1200 + 1200 \\ &= 2400\end{aligned}$$

$$\begin{aligned}\text{Average speed} &= \frac{\text{distance}}{\text{time}} \\ &= \frac{2400}{5} \\ &= 480\end{aligned}$$

The average speed is 480 km/h.

Example 4

A brick is dropped from the top of a 78.4 m building. Its height (in metres) after time t (in seconds) is $h = 78.4 - 4.9t^2$ (until it hits the ground). Determine the following:

- What is the brick's velocity after 1 s and after 2 s?
- When will the brick hit the ground?
- What is the brick's velocity just before it hits the ground?

Solution

The brick's position is defined as $h = 78.4 - 4.9t^2$.

$$\begin{aligned} \therefore\ v &= \frac{dh}{dt} \\ &= -9.8t \end{aligned}$$

Remember: The negative indicates that the brick is travelling downward.

When $t = 1$ s, $v = -9.8$ m/s

When $t = 2$ s, $v = -9.8(2)$
$= -19.6$ m/s

The brick will hit the ground when $h = 0$.

$$\begin{aligned} 0 &= 78.4 - 4.9t^2 \\ t^2 &= \frac{78.4}{4.9} \\ t &= 4 \end{aligned}$$

Since $t > 0$, t cannot be negative.

The brick will hit the ground after 4 s.

Its velocity just before it hits the ground is found by evaluating the velocity function at the time it hits the ground.

$$\begin{aligned} v(t) &= -9.8t \\ v(4) &= -9.8(4) \\ &= -39.2 \text{ m/s} \end{aligned}$$

The negative indicates downward motion.

The falling brick will hit the ground at 39.2 m/s.

You can try these questions.

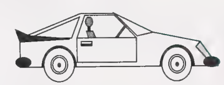
1. City A is 150 km from City B, and City B is 250 km from City C. The three cities are in a straight line as shown in the diagram. A car leaves City A at 8:00 A.M. and arrives at City B at 9:30 A.M. After a quick stopover, the same car leaves City B at 10:00 A.M. It arrives at City C at 12:30 p.m. It leaves City C at 1:00 P.M. and returns to City B at 4:00 P.M.

City A 150 km City B 250 km City C

   a. What is the car's displacement?
   b. What is the average velocity?
   c. What is the average speed?

2. A car is driven along a straight highway for three hours at 90 km/h and then for four hours at 100 km/h.

   a. What is its displacement?
   b. What is the average velocity?



3. An object is moving upward away from a fixed point. The position (in metres) of the object with respect to the fixed point is a function of time (in seconds) given by $h = -5t^2 + 45t + 25$, where $t \geq 0$.

   a. What is the velocity at any time?
   b. What is the velocity at $t = 1$?
   c. Find t when the object reaches its maximum displacement.
   d. What is its maximum displacement?
   e. What is the average velocity from $t = 0$ to $t = 2$?

4. An object is moving in a straight line from a fixed point. The displacement (in metres), which is a function of time (in seconds), is given by $s = t^3 - 9t^2 + 24t$.

   a. Find the average velocity from $t = 1$ to $t = 4$.
   b. What is the velocity at any time?
   c. What is the velocity when $t = 3$?
   d. Find t when s is a minimum.
   e. What is the minimum s?
   f. Draw a diagram to represent the motion of the particle.

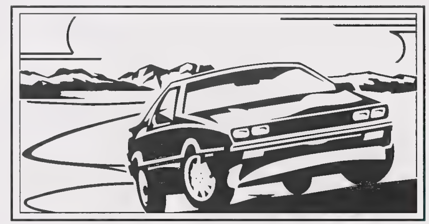
Check your answers by turning to the Appendix.

You should now be able to distinguish between distance and displacement, between speed and velocity, and between average velocity and instantaneous velocity.

Activity 2: Acceleration

Suppose you are driving a car and you wish to drive at a velocity of 50 km/h; the car in front of you is only driving at a velocity of 40 km/h. You step on the gas to increase the velocity of your car to 80 km/h in order to pass. You have accelerated your car.

Acceleration is the rate at which the velocity changes per unit time. If you increase the velocity of your car from 50 km/h to 80 km/h in three seconds, then the acceleration is $\frac{80-50}{3}$ km/h/s $= 10$ km/h/s. It can be written as 10 km/h/s. Acceleration is the rate of change of velocity.



$$\therefore \textbf{Average acceleration} = \frac{\text{corresponding change in velocity}}{\text{change in time}}$$

$$a_{ave} = \frac{\Delta v}{\Delta t}$$

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Again, make the change in time (Δt) very small. When Δt approaches zero, the acceleration becomes the acceleration at a given time. It is called **instantaneous acceleration** and is denoted by the symbol a.

Thus, $a = \lim\limits_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t}$.

Since $a = \lim\limits_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ is the derivative of v with respect to t, $a = \frac{dv}{dt}$.

But $v = \dfrac{ds}{dt}$.

Therefore, $a = \frac{d}{dt}\left(\frac{ds}{dt}\right)$, which is the second derivative of s with respect to t.

Review Module 3 for techniques for finding higher-order derivatives.

$\frac{d}{dt}\left(\frac{ds}{dt}\right)$ can be written as $\frac{d^2 s}{dt^2}$.

Therefore, $a = \dfrac{d^2 s}{dt^2} = \dfrac{dv}{dt}$.

If s is measured in metres and t is measured in seconds, then the units for acceleration are $\mathrm{m/s}^2$.

Caution: Notice that the expression **with respect to *t*** is emphasized and the symbol $\frac{d}{dt}$ is used. Always specify the variable being **differentiated**. (Differentiation is the process of taking the derivative.) D standing alone with no variable specified is meaningless. $\frac{d}{dt}\left(3t^2\right) = 6t$, but $\frac{d}{dx}\left(3t^2\right)$ cannot be calculated unless there is some way of expressing t in terms of x.

$\frac{d^2}{dt^2}$ does not mean $\frac{d}{dt}$ **squared**. It means **take the second derivative of**.

Look at the following examples.

Example 1

The velocity (in m/s) of an object is given by $v = 3t^2 - 5t + 8$, where $t \geq 0$. Find the acceleration at $t = 3$.

Solution

$v = 3t^2 - 5t + 8$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 6t - 5 \end{aligned}$$

When $t = 3$,
$$\begin{aligned} a &= 6(3) - 5 \\ &= 18 - 5 \\ &= 13 \end{aligned}$$

The acceleration of the object is $13\ \mathrm{m/s}^2$ when $t = 3$.

Example 2

A particle is moving in a straight line. The distance from a fixed point is given by $s = t^3 - 5t^2 + 7t + 9$, where $t \geq 0$.

- Determine the velocity and acceleration at any time.
- Determine the turning points; that is, the points where the velocity is zero.
- Draw the graph of $s = t^3 - 5t^2 + 7t + 9$.
- Describe the direction of the motion at $t = 0$, $t = 1$, $t = 2$, $t = 2\frac{1}{3}$, and $t = 3$ s.
- Draw the graphs of v versus t and a versus t. Describe both v and a.

Solution

$$v = \frac{ds}{dt} = 3t^2 - 10t + 7, \text{ where } t \geq 0$$

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = 6t - 10, \text{ where } t \geq 0$$

The velocity at any time is $3t^2 - 10t + 7$.
The acceleration at any time is $6t - 10$.

$$3t^2 - 10t + 7 = 0$$
$$(3t - 7)(t - 1) = 0$$
$$3t - 7 = 0 \quad \text{or} \quad t - 1 = 0$$
$$t = \frac{7}{3} \text{ or } 2\frac{1}{3} \qquad t = 1$$

When $t = 1$, $s = (1)^3 - 5(1)^2 + 7(1) + 9$
$= 12$

When $t = \frac{7}{3}$, $s = \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right) + 9$

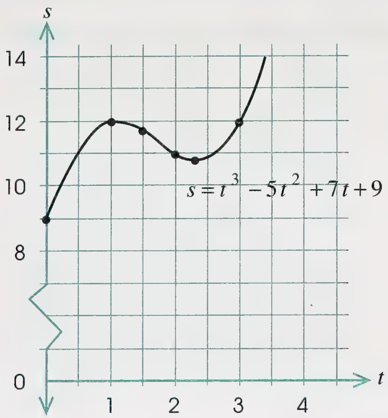
$$= \frac{343}{27} - \frac{245}{9} + \frac{49}{3} + 9$$
$$= \frac{343}{27} - \frac{735}{27} + \frac{441}{27} + \frac{243}{27}$$
$$= \frac{292}{27} \text{ or } 10\frac{22}{27}$$

Therefore, $(1, 12)$ and $\left(\frac{7}{3}, 10\frac{22}{27}\right)$ are the turning points.

To draw the graph of $s = t^3 - 5t^2 + 7t + 9$, use a table of values. The following is a sample calculation.

$$s = (2)^3 - 5(2)^2 + 7(2) + 9$$
$$= 8 - 20 + 14 + 9$$
$$= 11$$

t	s
0	9
1	12
$1\frac{1}{2}$	$11\frac{5}{8}$
2	11
$2\frac{1}{3}$	$10\frac{22}{27}$
3	12



When $t = 0$, $v = 0 + 0 + 7$

$= 7$

Therefore, the direction of motion is positive.

When $t = 1$, $v = 3 - 10 + 7$

$= 0$

Therefore, the particle stops.

When $t = 2$, $v = 3(2)^2 - 10(2) + 7$

$= -1$

Therefore, the direction of motion is negative (moving back toward the starting point).

When $t = 2\frac{1}{3}$, $v = 3\left(\frac{7}{3}\right)^2 - 10\left(\frac{7}{3}\right) + 7$

$= 0$

The particle stops.

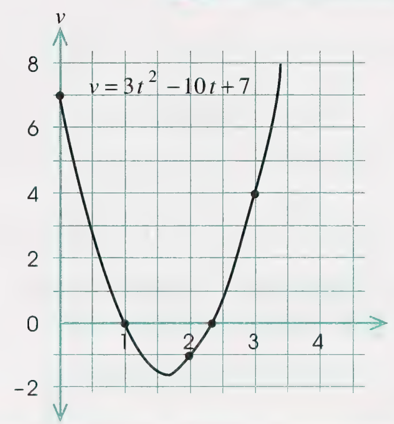
When $t = 3$, $v = 3(3)^2 - 10(3) + 7$

$= 4$

The direction of motion is positive.

To draw the graph of v versus t $\left(v = 3t^2 - 10t + 7\right)$, use a table of values.

t	0	1	2	$2\frac{1}{3}$	3
v	7	0	-1	0	4



$$\frac{dv}{dt} = 6t - 10 = 0$$

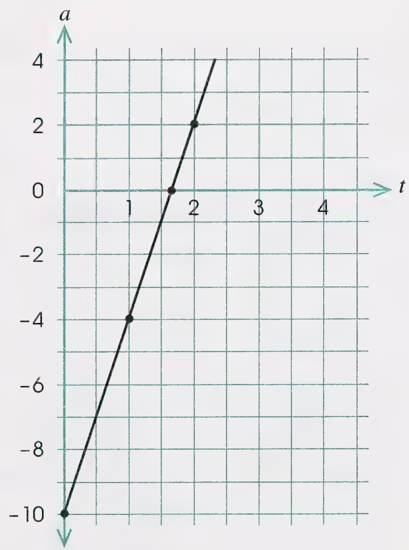
$$t = \frac{10}{6}$$

$$= 1\frac{2}{3}$$

The velocity is decreasing from $t=0$ to $t=1$. When $t=1$, the particle stops and the velocity is zero. From $t>1$ to $t<2\frac{1}{3}$, the particle is moving in the opposite direction. It stops again when $t=2\frac{1}{3}$. The particle turns around and moves in a positive direction at $t>2\frac{1}{3}$. The velocity is positive and increasing after that.

To draw the graph of a versus t $(a=6t-10)$, use a table of values.

t	a
0	−10
1	−4
$1\frac{2}{3}$	0
2	2



The acceleration is increasing from negative to positive, or, for example, from deceleration to acceleration at a constant rate. When a is negative, the velocity decreases. When a is positive, the velocity increases.

Sometimes more complex functions can be used to define a velocity or displacement function, but the basic principles remain the same.

Example 3

A particle moves in a straight line. The displacement (in metres) is given by the following function.

$$s=\frac{t^2}{2+t}, \text{ where } t\geq 0$$

Determine the acceleration at the instant when the velocity is $\frac{3}{4}$ m/s.

Solution

$$\therefore\ v=\frac{ds}{dt}=\frac{(2+t)D_t\left(t^2\right)-\left(t^2\right)D_t(2+t)}{(2+t)^2}$$

quotient rule

$$=\frac{(2+t)(2t)-t^2}{(2+t)^2}$$

$$=\frac{4t+2t^2-t^2}{(2+t)^2}$$

$$=\frac{t^2+4t}{(2+t)^2}$$

$$
\begin{aligned}
\therefore\ a = \frac{dv}{dt} &= \frac{(2+t)^2 D_t\left(t^2+4t\right)-\left(t^2+4t\right)D_t(2+t)^2}{(2+t)^4} \\
&= \frac{(2+t)^2(2t+4)-\left(t^2+4t\right)(2)(2+t)}{(2+t)^4} \\
&= \frac{(2+t)^2(2)(2+t)-2(2+t)\left(t^2+4t\right)}{(2+t)^4} \\
&= \frac{2(2+t)\left[\left(4+4t+t^2\right)-\left(t^2+4t\right)\right]}{(2+t)^4} \\
&= \frac{2(2+t)(4)}{(2+t)^4} \\
&= \frac{8}{(2+t)^3}
\end{aligned}
$$

You are given $v = \frac{3}{4}$ m/s.

$$
\begin{aligned}
\therefore\ \frac{t^2+4t}{(2+t)^2} &= \frac{3}{4} \\
4\left(t^2+4t\right) &= 3(2+t)^2 \\
4t^2+16t &= 12+12t+3t^2 \\
t^2+4t-12 &= 0 \\
(t+6)(t-2) &= 0
\end{aligned}
$$

$t+6=0$ or $t-2=0$

$t=-6$ $\quad$ $t=2$

Since $t \geq 0$, $t \neq -6$.

$$
\begin{aligned}
\text{When } t=2,\ a &= \frac{8}{(2+t)^3} \\
&= \frac{8}{(2+2)^3} \\
&= \frac{1}{8}
\end{aligned}
$$

The acceleration of the particle is $\frac{1}{8}$ m/s^2 when the velocity is $\frac{3}{4}$ m/s.

You can now practise the concepts of this activity.

1. In each of these questions, the velocity of a particle is given. Find the acceleration at any time and at the particular times required.

a. $v = 2t-3t^2$
Find a when $t=4$.

b. $v = (t+3)^2$
Find a when $t=3$.

c. $v = 5t^3-2t$
Find a when $t=2$.

d. $v = \dfrac{10t}{3+t^2}$
Find a when $t=1$.

2. An object is moving in a straight line. The displacement (in metres) from a fixed point is given by $s = t^2-5t+4$, where $t \geq 0$.

a. Find the velocity and acceleration at any time.

b. Determine the turning point; that is, the point where the velocity is zero.

c. Draw the graph of the function $s(t)$.

d. Describe the direction of the motion at $t = 1$, $t = \frac{5}{2}$, and $t = 3$.

e. Draw the graph of v versus t and describe the velocity.

f. Draw the graph of a versus t and describe the acceleration.

3. An object is moving in a straight line. The displacement (in metres) from a fixed point is given by $s = 2t^3 - 15t^2 + 24t + 8$, where $t \geq 0$.

a. Find the velocity and acceleration at any time.

b. Determine the turning points; that is, the points where the velocity is zero.

c. Draw the graph of the function $s(t)$.

d. Describe the direction of the motion at $t = 0$, $t = 1$, $t = 3$, $t = 4$, and $t = 5$.

e. Draw the graph of v as a function of t and describe the velocity.

f. Draw the graph of a as a function of t and describe the acceleration.

4. A particle is moving in a straight line. The position of the particle (in metres) from a fixed point is given by $s = 3t^3 - 3t^2 + t + 9$, where $t \geq 0$. Determine the acceleration when the velocity is 4 m/s.

5. A particle is moving in a straight line. Its position (in metres) from a fixed point is given by $s = \frac{t^2}{3+t}$, where $t \geq 0$. Determine the acceleration when the velocity is $\frac{3}{4}$ m/s.

Check your answers by turning to the Appendix.

If you look back at Example 4 of Activity 1, the displacement function for calculating the height of a brick dropped from a rooftop was $h = 78.4 - 4.9t^2$. Taking the derivative gave you the velocity function $v(t) = -9.8t$. If you take a second derivative to find acceleration, you get $a(t) = -9.8$, a constant. This constant is called **acceleration due to gravity**.

If a physical object is released at or near Earth's surface, it has been calculated that the object will accelerate toward the centre of Earth at 9.8 m/s^2. This acceleration is caused by gravity. This acceleration due to gravity is the same for all objects regardless of their size or mass. A feather and a brick will have the same acceleration in Earth's gravitational field.

If an object is thrown upward, its upward velocity will decrease because of the downward pull (acceleration due to gravity). Its upward velocity will decrease until $v(t)=0$, where it reaches its maximum height. Then, the object will descend because of the steady downward pull of 9.8 m/s^2, and its downward velocity will increase with time. That is, the ball slows down as it rises and speeds up as it falls.

In general, a negative acceleration indicates that velocity is decreasing (a tangent with a negative slope), and a positive acceleration indicates that the velocity is increasing (a tangent with a positive slope).

Example 4

If a ball is thrown upward with a velocity of 5 m/s from the top of a 90 m tower, then the distance (in metres) above the ground after t seconds is $s=90+5t-5t^2$. When does the ball reach its maximum height? How long does it take for the ball to reach the ground? Determine the velocity just before the ball hits the ground.

Solution

The maximum height occurs when $v(t)=0$.

$$\frac{ds}{dt}=v(t)=5-10t=0$$
$$t=\frac{1}{2}$$

The ball will reach its maximum height in 0.5 s.

The ball at ground level means $s=0$.

$$\begin{aligned}90+5t-5t^2 &= 0\\ 18+t-t^2 &= 0\\ t^2-t-18 &= 0\end{aligned}$$

Use the quadratic formula to solve for t.

$$t=\frac{-(-1)\pm\sqrt{1-4(-18)(1)}}{2(1)}$$
$$\therefore\ t=\frac{1+\sqrt{73}}{2}$$
$$\doteq 4.8$$

Time t must be greater than zero.

The ball hits the ground at approximately 4.8 s.

The velocity when the ball hits the ground is found at $v(4.8)$.

$$\begin{aligned}v(t)&=5-10t\\ v(4.8)&\doteq 5-10(4.8)\\ &\doteq 5-48\\ &\doteq -43\end{aligned}$$

The ball hits the ground at approximately -43 m/s.

Example 5

The position function of a particle moving in a straight line is given as $s(t)=t^3-3t^2+5$. Its position (in centimetres) is relative to a fixed point. Determine the following:

- initial position
- intitial velocity
- average acceleration from $t=1$ s and $t=3$ s
- instantaneous acceleration at $t=2$ s
- pattern of motion, represented in a diagram

Solution

The initial position is found at $t=0$.

$$\begin{aligned} s(t)&=t^3-3t^2+5 \\ s(0)&=(0)^3-3(0)^2+5 \\ &=5 \end{aligned}$$

The particle, at $t=0$, is 5 cm to the right of the origin.

The initial velocity is found at $t=0$.

$$\begin{aligned} v(t)&=3t^2-6t \\ v(0)&=3(0)^2-6(0) \\ &=0 \end{aligned}$$

The particle is not moving at $t=0$.

To determine average acceleration, you must evaluate the velocity function at $t=1$ and $t=3$.

$$\begin{aligned} v(1)&=3(1)^2-6(1) & v(3)&=3(3)^2-6(3) \\ &=3-6 & &=27-18 \\ &=-3 & &=9 \end{aligned}$$

$$\begin{aligned} a_{ave}&=\frac{\Delta v}{\Delta t} \\ &=\frac{v(3)-v(1)}{3-1} \\ &=\frac{9-(-3)}{2} \\ &=\frac{12}{2} \\ &=6 \end{aligned}$$

The average acceleration from $t=1$ s to $t=3$ s is 6 m/s^2. This means that the particle is speeding up.

The instantaneous acceleration at $t=2$ s is found by evaluating the acceleration function at 2.

$$\begin{aligned} s(t)&=t^3-3t^2+5 \\ v(t)&=\frac{ds}{dt}=3t^2-6t \\ a(t)&=\frac{dv}{dt}=6t-6 \end{aligned}$$

$$a(2) = 6(2) - 6$$
$$= 12 - 6$$
$$= 6$$

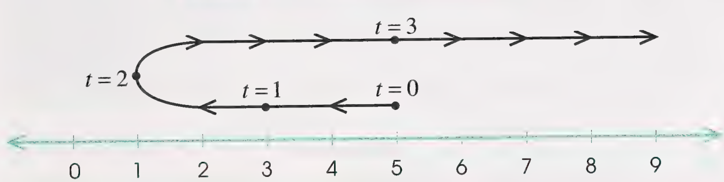
The instantaneous acceleration at $t = 2$ s is 6 m/s^2.

To represent the particle's motion in a diagram, you must evaluate the position function at various times and find any turning points by evaluating the velocity to 0.

$$v(t) = 3t^2 - 6t = 0$$
$$t(3t - 6) = 0$$
$$t = 0 \text{ or } 3t - 6 = 0$$
$$t = 2$$

There is a turning point at $t = 2$.

t	0	1	2	3	4
$s(t)$	5	3	1	5	21



Since the particle's initial position is 5 cm to the right of the origin and it moves to the left in the first 2 s, its velocity must be negative beween 0 s and 2 s.

Check

$$v(1) = 3(1)^2 - 6(1)$$
$$= 3 - 6$$
$$= -3 \text{ m/s}$$

Example 6

An object is moving in a straight line and its position, relative to a fixed point, is defined by $s(t) = t^3 - 4t^2 + 12t$, where s is in metres and t is in seconds. Analyse its motion when $t = 1$ s.

Solution

$$s(t) = t^3 - 4t^2 + 12t$$
$$s(1) = (1)^3 - 4(1)^2 + 12(1)$$
$$= 1 - 4 + 12$$
$$= 9$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 8t + 12$$
$$v(1) = 3(1)^2 - 8(1) + 12$$
$$= 3 - 8 + 12$$
$$= 7$$

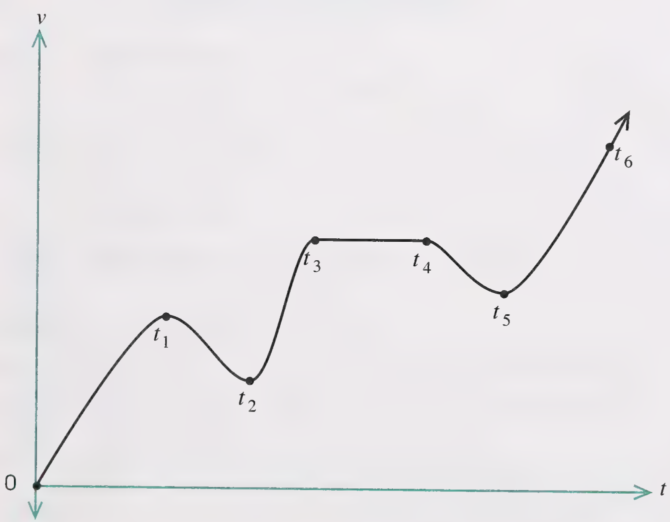
$$a(t) = \frac{dv}{dt} = 6t - 8$$
$$a(1) = 6(1) - 8$$
$$= 6 - 8$$
$$= -2$$

At $t = 1$ s, the object's position is 9 m; its velocity is 7 m/s; and its acceleration is -2 m/s^2.

The object can be found 9 m to the right of the origin. Also, since its velocity is positive, it continues moving to the right. The acceleration and velocity have opposite signs. Therefore, it is slowing down.

Watch the video titled *Motion: Distance, Velocity and Accleration*, from the *Catch 31* series, ACCESS Network. This program should help you understand the applications of velocity and acceleration. This program is available from the Learning Resources Distributing Centre.

6.



Using the preceding graph of a velocity function, state whether the acceleration is positive, negative, or zero for each of the following intervals.

a. 0 to t_1 **b.** t_1 to t_2 **c.** t_2 to t_3
d. t_3 to t_4 **e.** t_4 to t_5 **f.** t_5 to t_6

7. A particle's position is defined by the function $s(t) = 2t^3 - 24t$, where $t \geq 0$, position s is in centimetres, and time t is in seconds. Find the acceleration at the moment when the velocity is 0 cm/s.

8. An object is moving along a path $s(t) = t^3 - 3t^2 - 12t$, where s is in metres and t is in seconds.

a. At what time is the acceleration zero?

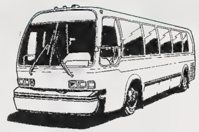
b. What is the position of the object and its velocity at the time $a = 0$?

c. Describe its motion at the time $a = 0$.

9. The position function of a particle is given as $s(t) = t^4 - 10t^3 + 36t^2 + 10t + 12$, where $t \geq 0$. Find the intervals for which the acceleration is positive and negative.

10. State the sign of the acceleration and velocity if an object is 10 m to the left of the origin, is moving away from the origin, and is speeding up.

11. A bus is travelling 108 km/h and the brakes are fully applied. If braking produces a constant deceleration of 15 m/s^2, and the velocity function is $v(t) = -15t + 30$, how long does it take the bus to stop?



Check your answers by turning to the Appendix.

In the first part of this activity you dealt with motion of particles given their displacement or velocity as a function of time.

In certain instances a relationship between displacement and velocity may be given by an equation like $3s^2 + 5v^2 = 28$.

What is the acceleration in this case?

Relations like this and others can be dealt with using an adaption of the chain rule called **implicit differentiation**. The method of implicit differentiation was discussed in Module 3, Section 2, Activity 8.

Look at an example.

Example 7

If $3s^2 + 5v^2 = 28$, where s is displacement and v is velocity, find the acceleration a.

Solution

Since s, v, and a are all functions of t, you can differentiate any of them with respect to t.

Begin by taking the derivative of each term with respect to t.

$$3s^2 + 5v^2 = 28$$

$$\begin{aligned} \frac{d}{dt}\left(3s^2\right) + \frac{d}{dt}\left(5v^2\right) &= \frac{d}{dt}(28) \\ 6s\frac{ds}{dt} + 10v\frac{dv}{dt} &= 0 \end{aligned}$$

Since $\frac{ds}{dt} = v$ and $\frac{dv}{dt} = a$, then incorporate these into the equation.

$$\begin{aligned} 6sv + 10va &= 0 \\ v(6s + 10a) &= 0 \\ 6s + 10a &= 0 \\ a &= -\frac{6s}{10} \\ &= -\frac{3s}{5} \end{aligned}$$

For additional help, view the video titled *Functions Defined Implicitly*, from the *Catch 31* series, ACCESS Network. This video is available from the Learning Resources Distributing Centre.

Now you can use implicit differentiation to solve some other motion problems.

Example 8

The relation between the velocity and distance is given by $5v^2 = 40s + 200$, where s is the displacement from a fixed point and v is the velocity of a moving object. Find the acceleration.

Solution

$$5v^2 = 40s + 200$$

$$\frac{d}{dt}\left(5v^2\right) = \frac{d}{dt}(40s) + \frac{d}{dt}(200)$$

$$10v\frac{dv}{dt} = 40\frac{ds}{dt} + 0$$

Substitute $\frac{dv}{dt} = a$ and $\frac{ds}{dt} = v$ into the equation.

$$\begin{aligned} 10va &= 40v \\ a &= \frac{40v}{10v} \\ &= 4 \end{aligned}$$

When taking the derivative of an equation, it is important to know which symbol represents a constant.

12. An object is moving away from a fixed point. The relation between the velocity and the distance is given by $3v^2 = 18s + 300$, where s is the distance from a fixed point and v is the velocity. Find the acceleration.

13. An object is moving away from a fixed point. It has a mass m (in kilograms) and has moved a distance s (in metres). Its velocity is v. The total energy is given by $E = \frac{1}{2}mv^2 + mps$. If E, m, and p are constants, find the acceleration a.

14. A gun is fired. The bullet travelled a distance s in the barrel of the gun. The velocity of the bullet is given by $v = \frac{6000s}{6+s}$. Find the acceleration in terms of s.

15. The relation between the velocity v and time t is given by $\frac{1}{v} + \frac{1}{3} = 4t$. Prove that the acceleration is $-4v^2$.

16. Find $\frac{dy}{dx}$ if $\sqrt{y^2 - 2} = xy$.

17. Find $\frac{dy}{dx}$ if $x^3y - 4y^2 = 3$.

Check your answers by turning to the Appendix.

You should now be able to find and interpret the acceleration of an object from either its position or velocity function. Step on the accelerator and drive on to the next activity.

Activity 3: Applications in Science

In this section, you have been using the derivative to determine the rate of change of one variable with respect to another. You have studied rates of change (velocity and acceleration) in straight line motion and falling objects as one application in physics. Other applications are found in the sciences.

Example 1

A spherical balloon is being inflated. The volume is changing with respect to the balloon's radius. Find the rate of change of the balloon's volume when its radius is 15 cm.

Solution

The formula for the volume of a sphere with respect to radius is $V = \frac{4\pi r^3}{3}$.

The rate of change of volume with respect to the radius is as follows:

$$\frac{dV}{dr} = 4\pi r^2$$

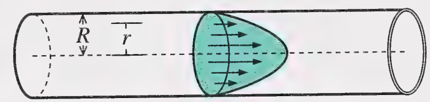
$$\text{When } r = 15,\ \frac{dV}{dr} = 4\pi(15)^2 = 900\pi$$

Therefore, the rate of change of the volume with respect to radius at $r = 15$ cm is $900\ \text{cm}^3/\text{cm}$ or 900 ml/cm.

The following example determines the **velocity gradient** of a blood vessel.

Example 2

The blood flow through a blood vessel, assumed to be a cylindrical shape with radius R and the length ℓ, is subject to friction along the walls of the vessel. Because of this friction, the velocity of the blood is the greatest at the central axis of the tube and decreases as it gets closer to the walls. As the distance r from the axis increases, velocity decreases, becoming 0 at the wall.



The relationship between v and r is given by the Law of Laminar Flow discovered in 1840 by the French physician Poiseuille. The law states the following:

$$v = \frac{P}{4\eta\ell}\left(R^2 - r^2\right)$$

In this formula, η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P, η, and ℓ are constants, then v is a function of r.

A typical human artery has a radius of 0.008 cm, a blood viscosity of 0.027 when $\ell = 2$ cm, and $P = 4000$ dynes/cm^2. Find the velocity gradient (rate of change of v with respect to r) when $r = 0.003$ cm.

Solution

$$\frac{dv}{dr}=\frac{P}{4\eta\ell}(0-2r)=-\frac{Pr}{2\eta\ell}$$

When $r=0.003$, $\dfrac{dv}{dr}=-\dfrac{4000(0.003)}{2(0.027)(2)}$

$$\doteq -111.1(\text{cm/s})/\text{cm}$$

You can find this answer by using a calculator.

(4 0 0 0 +/_ × 0 • 0 0 3) ÷ (2 × 0 • 0 2 7 × 2) =

-111.1111111

The velocity gradient is $-111.1\ (\text{cm/s})/\text{cm}$.

Example 3

An object is dropped from a window. The total energy E (in joules) is given by $E=\frac{1}{2}mv^2+mgs$, where m is the mass of the object (in kilograms), g is the acceleration due to gravity $\left(-9.8\ \text{m/s}^2\right)$, v is the velocity (in metres per second), and s is the distance from the ground (in metres). If E is a constant, find the acceleration a.

Solution

$$\frac{d(E)}{dt}=\frac{d\left(\frac{1}{2}mv^2\right)}{dt}+\frac{d(mgs)}{dt}$$

Since E is a constant, $\frac{dE}{dt}=0$.
Since m is the mass of the object, it is a constant.

Since g is always equal to $-9.8\ \text{m/s}^2$, it is a constant.

$$\begin{aligned}\therefore\ 0&=\frac{1}{2}m\frac{dv^2}{dt}+mg\frac{ds}{dt}\\ 0&=\frac{1}{2}m(2v)\frac{dv}{dt}+mg\frac{ds}{dt}\\ &=mv(a)+mg(v)\end{aligned}$$

$$\begin{aligned}a&=-\frac{mgv}{mv}\\ &=-g\end{aligned}$$

Acceleration is equal to the negative value of gravity.

Try these questions.

1. Find the rate of change of the area of a circle with respect to its radius when its diameter is 12 cm.

2. Find the rate of change of the volume of a cube with respect to its edge e when $e = 5$ mm.

3. Find the velocity gradient for blood flowing through an artery under the same conditions as Example 2 when $r = 0.004$ cm.

4. The population growth of a colony of bacteria is given as $n = 250 + 10t + t^2$. Find the growth rate after three hours.

5. The cost of producing x units of an item is given as $C(x) = 5000 + 5x + 0.05x^2$. Compare the marginal cost of producing 101 items with the cost of producing 100 items.

Check your answers by turning to the Appendix.

The interpretation of the derivative as a rate is crucial to modelling real-world situations in the natural sciences.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

To work with the material in this section expertly, it is helpful to follow these steps:

Step 1: After reading the problem carefully, decide if it is asking for an average rate of change or an instantaneous rate of change. For instance, in problems concerned with average velocity, a time interval will be given and problems concerned with instantaneous velocity will cite one specific time.

Step 2: Make careful note of the algebraic signs of the quantities found. Signs indicate direction of motion: up and right are positive and left and down are negative. Signs also indicate direction of change. For an object to speed up, its instantaneous velocity and acceleration must be in the same direction (both positive or both negative); for an object to slow down, instantaneous velocity and accleration are in opposite directions.

Step 3: Express all answers with appropriate units. The position of an object is given as a unit of length, velocity is measured as a rate unit (e.g., cm/s), and acceleration is stated with an extended rate unit $\left(\text{e.g., cm/s}^2\right)$.

The average rate of change is different from the instantaneous rate of change.

If $y = f(x)$,

Average Rate of Change	Instantaneous Rate of Change
$\frac{\Delta y}{\Delta x}$ or $\frac{\Delta f}{\Delta x}$	$\frac{dy}{dx}$ or $\frac{df}{dx}$
slope of a secant of $y = f(x)$	slope of a tangent to $y = f(x)$

The following are the specific formulas used for average and instantaneous rates of change for velocity and acceleration and their geometric interpretations if $s = \text{position}$, $t = \text{time}$, $v = \text{velocity}$, and $a = \text{acceleration}$.

Average Velocity	Instantaneous Velocity
$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$	$v(t) = \frac{ds}{dt}$
slope of the secant of the position-time graph	slope of a tangent to the position-time graph

Average Acceleration	Instantaneous Acceleration
$a_{ave} = \frac{\Delta v}{\Delta t} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$	$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$
slope of the secant of the velocity-time graph	slope of the tangent to the velocity-time graph

When velocity is zero, the object reaches its local maximum or minimum and is represented geometrically by a tangent that is a horizontal line. When acceleration is zero, the object reaches its maximum or minimum velocity, indicating that when $a = 0$, velocity is constant.

Example 1

If $s=t^3-6t^2+9t+24$, where $t \geq 0$, then the instantaneous velocity at any time is $v=\frac{ds}{dt}=3t^2-12t+9$.

The instantaneous velocity at $t=2$ is as follows:

$$\begin{aligned} v &= 3(2)^2-12(2)+9 \\ &= -3 \end{aligned}$$

The acceleration at any time is $a=\frac{dv}{dt}=6t-12$.

The acceleration at $t=3$ is $a=6(3)-12=6$.

The average velocity in the interval from $t=2$ to $t=5$ is as follows:

$$\text{When } t=2,\ s=(2)^3-6(2)^2+9(2)+24$$
$$=8-24+18+24$$
$$=26$$

$$\text{When } t=5,\ s=(5)^3-6(5)^2+9(5)+24$$
$$=125-150+45+24$$
$$=44$$

$$\begin{aligned} \text{Average velocity} &= \frac{44-26}{5-2} \\ &= \frac{18}{3} \\ &= 6 \end{aligned}$$

The distance has a maximum or minimum when $\frac{ds}{dt}=v=0$.

$$\begin{aligned} 3t^2-12t+9&=0 \\ t^2-4t+3&=0 \\ (t-1)(t-3)&=0 \\ t-1=0 \quad &\text{or} \quad t-3=0 \\ t=1 \quad & \quad\quad t=3 \end{aligned}$$

$$\text{When } t=1,\ s=(1)^3-6(1)^2+9(1)+24$$
$$=28$$

$$\begin{aligned} a=\frac{d^2s}{dt^2} &= 6(1)-12 \\ &= -6 \quad (\text{negative}) \end{aligned}$$

Therefore, 28 is a maximum distance.

$$\text{When } t=3,\ s=(3)^3-6(3)^2+9(3)+24$$
$$=24$$

$$a = \frac{d^2 s}{dt^2} = 6(3) - 12$$
$$= 6 \quad (\text{positive})$$

Therefore, 24 is a minimum distance.

The velocity has a maximum or minimum when $\frac{dv}{dt} = 0$.

$$a = \frac{dv}{dt} = 6t - 12 = 0$$
$$6t = 12$$
$$t = 2$$

When $t = 2$, $v = 3(2)^2 - 12(2) + 9$

$$= 12 - 24 + 9$$
$$= -3$$

$$\frac{d^2 v}{dt^2} = 6 \quad (\text{positive})$$

Therefore, -3 is a minimum velocity.

Implicit differentiation can be used to solve motion problems. If a relation between s and v is given and you want to find the acceleration, which is $\frac{dv}{dt}$, implicit differention is the tool to use.

Example 2

If $v^2 = 4s^2 + 200$, where v is the velocity of a moving object and s is the distance from a fixed point, find the acceleration at any time.

Solution

$$v^2 = 4s^2 + 200$$
$$\frac{d}{dt}\left(v^2\right) = \frac{d}{dt}\left(4s^2\right) + \frac{d}{dt}(200)$$
$$2v\frac{dv}{dt} = 8s\frac{ds}{dt} + 0$$

$$\frac{dv}{dt} = a \text{ and } \frac{ds}{dt} = v$$

$$\therefore\ 2va = 8sv$$
$$a = \frac{8sv}{2v}$$
$$= 4s$$

The acceleration at any time is $a = 4s$.

1. If $s = 2t^3 - 21t^2 + 60t + 80$, where s is the distance (in metres) and t is the time (in seconds), determine the following:

   a. the instantaneous velocity at $t = 2$
   b. the instantaneous acceleration at $t = 3$
   c. the average velocity in the interval from $t = 1$ to $t = 3$
   d. the maximum or minimum velocity (if any)

2. If $x^2 + 2xy + 4y^2 = 3$, find $\frac{dy}{dx}$.

3. If $3v^2 = 5s + 1000$, where v is the velocity of a moving object and s is the distance from a fixed point, find the acceleration at any time.

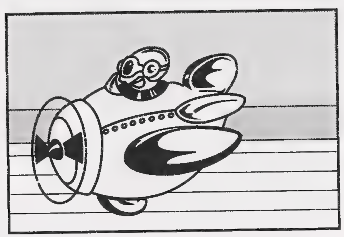
Check your answers by turning to the Appendix.

Enrichment

When given a velocity function expressed in terms of the displacement s, the acceleration function is found by differentiating the velocity function with respect to time. In order to express the acceleration function in terms of s, all derivatives formed of the type $\frac{ds}{dt}$ must be replaced by the original velocity function, as illustrated in the following example.

Example

The velocity v of a wind-up toy (in metres per second) is given by $v = \frac{300s}{3+s}$, where s is the length of the spring. Find the acceleration in terms of the length s.



Solution

$$\frac{dv}{dt} = \frac{(3+s)D_t(300s) - (300s)D_t(3+s)}{(3+s)^2}$$

$$\frac{dv}{dt} = \frac{(3+s)(300D_ts) - (300s)D_ts}{(3+s)^2}$$

$$\frac{ds}{dt} = v \text{ and } \frac{dv}{dt} = a$$

$$a = \frac{(3+s)(300v) - 300sv}{(3+s)^2}$$

By factoring you get $a = \dfrac{300v(3+s-s)}{(3+s)^2}$

$$= \frac{900v}{(3+s)^2}$$

$$v = \frac{300s}{3+s} \quad \text{(from the original equation)}$$

$$\therefore a = \frac{900}{(3+s)^2} \times \frac{300s}{(3+s)}$$

$$= \frac{270\,000s}{(3+s)^3}$$

The acceleration, in terms of s, is $a = \frac{270\,000s}{(3+s)^3}$.

Now try the following.

1. If $s^2 + 25v^2 = 100$ defines a relation between distance s and velocity v, find a relation between acceleration a and distance s.

2. The velocity v is given by $v = \frac{1-3s^2}{3s^2-2}$, where s is the displacement. Find the acceleration a in terms of s.

Check your answers by turning to the Appendix.

Conclusion

The activities in this section alerted you to another way to use the abstract mathematical concept of the derivative. You determined derivatives in Module 3 and now have interpreted them in practical applications in the sciences.

A diver's position when diving from the diving platform is a function of time. The velocity and acceleration at any time t can be found by successive applications of the derivative. Velocity, acceleration, change in volume, blood flow, energy, and marginal cost are all special cases of the derivative. The sciences provide many more opportunities for this application. You will study more in the next section.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 3: Related Rates

Have you ever had the feeling that time was running out? When you are playing a board game that has a time limit, does the time seem to go faster as you get closer to the deadline? Does it look like the sand is draining faster the closer it gets to the bottom of the timer? As there is a change in the volume of the sand, there also seems to be a change in the rate. This is only an illusion, since the hole that releases the sand in the timer doesn't change; the time in which it takes the sand to drain is constant. The change you do see taking place is a change in volume and a change in the depth of the sand in the top of the timer.

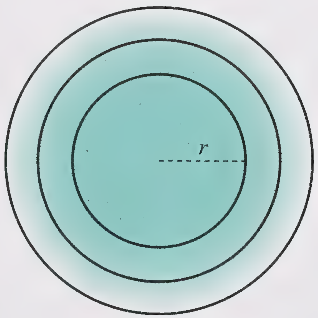
The rate at which the volume of the sand is changing depends on the depth of the funnel-shaped timer and the rate at which the level of the sand is changing. Finding these types of rates is called a related-rate problem. This kind of problem presents a situation in which one or more quantities are changing, and requires you to find the rate at which one of the quantities is changing. You will be using the chain rule (developed in Module 3) to help solve these problems.

In this section you will study some of the classic examples of related-rate problems. Using techniques you have studied in previous modules and sections, you will solve problems in which rates of change of some variables are dependent on each other.

Activity 1: Areas and Volumes

Relations in mathematics refer to a correspondence between two or more variables. This correspondence or rule is often expressed as an equation. If one of the quantities in the equation changes with time, then any other quantity that depends on it also changes with time.

For example, $C = 2\pi r$ expresses the relationship between the circumference of a circle and its radius. If the radius of a circle is increasing at a certain rate, what happens to the circumference of that circle? The circumference of the circle is also increasing.



The rate of change of a variable implies a change **with respect to time**. You have already dealt with questions that involve the rate of change of displacement (velocity), and the rate of change of velocity (acceleration). In that discussion, the rates of change at any time were determined by differentiating **with respect to time**. If you do the same here, you will find a relationship between the rates of change of the radius and the circumference of the circle.

Given $C = 2\pi r$, you may not know in detail how the dimensions of the region are changing with time, but you know that they do change with respect to time. So, in the case of the circle, you treat the radius and circumference as implicit functions of time, and therefore differentiate implicitly to determine the relationship between their rates of change.

According to the chain rule, if $y = f(x)$, then differentiating y with respect to time gives you the following:

$$\frac{dy}{dt} = \frac{dy}{dx} \bullet \frac{dx}{dt}$$

In like manner, given $C = 2\pi r$, when you differentiate with respect to time, $\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$.

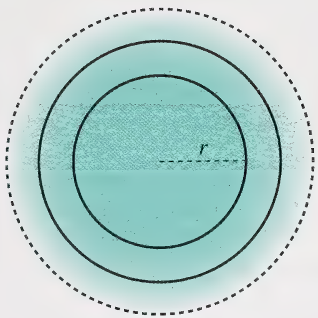
That is, $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$.

The relationship indicates that the rate of change of circumference **with respect to time** is equal to 2π times the rate of change of the radius **with respect to time**.

An example of the concept of **related rates** is the circumference of a circle changing when the radius changes.

Example 1

A circular ring is heated so that it expands. If the rate of increase of the radius is 0.01 cm/s, determine the rate at which the circumference is increasing.



Solution

The circumference of a circle is related to the radius by the following formula:

$$C = 2\pi r$$

Differentiating with respect to time, you get the following:

$$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$$
$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

You are given that the rate of change of the radius is 0.01 cm/s. Hence, $\frac{dr}{dt} = 0.01$ cm/s. Since the ring is expanding, the rate of change is positive.

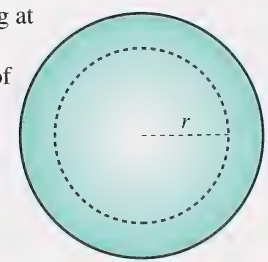
$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$
$$\frac{dC}{dt} = 2\pi(0.01)$$
$$\frac{dC}{dt} = 0.02\pi$$

The circumference of the ring is increasing at the rate of 0.02π cm/s.

The preceding example presents a relatively simple application of the concept of related rates. Now look at a more complex example.

Example 2

A circular disk is heated and then cooled. During the cooling process, the radius is found to be decreasing at the rate of 0.02 mm/s. At what rate is the area of the disk changing when the radius of the disk is 100 mm?



Solution

The area of the disk and its radius are related by the following formula:

$$A = \pi r^2$$

Differentiating with respect to time, you get the following:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi r^2\right)$$
$$= \frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt} \quad \text{(chain rule)}$$

That is, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.

You are given $\frac{dr}{dt} = -0.02$ mm/s and you are interested in the rate of change of area when $r = 100$ mm. Substitute these values into the equation.

As time increases, radius decreases; thus, $\frac{dr}{dt}$ is negative.

$$\frac{dA}{dt} = 2\pi(100)(-0.02)$$
$$\frac{dA}{dt} = -4\pi$$

The area is decreasing (the sign of $\frac{dA}{dt}$ is negative) at the rate of $4\pi \text{ mm}^2/\text{s}$.

A method for solving related-rate problems, suggested by the previous examples, follows:

Step 1: Determine a relationship between the variable(s) for which the rate(s) of change is (are) given and the variable for which the rate of change is to be found.

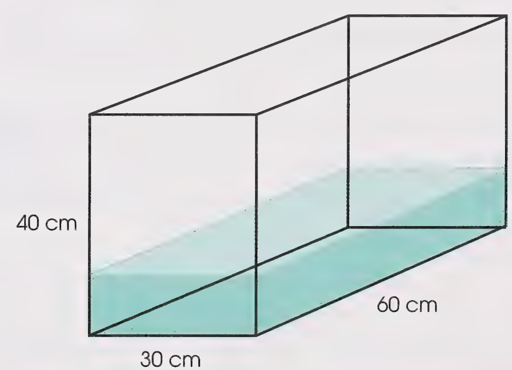
Step 2: Differentiate the equation with respect to time.

Step 3: Substitute the values given (or detemined by using the given values), and solve for the desired rate.

Apply this procedure to another example.

Example 3

Water is being poured into an aquarium that is 60 cm long, 30 cm wide, and 40 cm deep. Determine a relationship between the rate at which water is being poured and the rate at which the depth is increasing.



Solution

You are looking for a relation between the rate of change of **volume of water** and the rate of change of **depth of water**. As water is poured into the tank, it takes the shape of the container.

$$\therefore V_{water} = \ell_{water} \times w_{water} \times h_{water}$$

The 40 cm is not used in this problem, other than as a limit for h.

Of the three dimensions, the only one that varies is h_{water}.

$$\therefore V = 60 \times 30 \times h$$

$$= 1800h$$ (relates the volume of water to the depth of the water)

Differentiating with respect to time, you get $\frac{dV}{dt} = 1800\frac{dh}{dt}$.

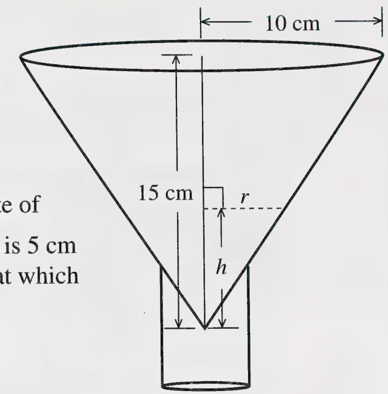
This is the relationship between the rates of change of volume and depth of water.

This example shows the importance of identifying variables and constants in the problem. If a quantity remains constant throughout the problem, its value can be substituted **before** you differentiate. If the quantity is variable, you cannot substitute any value given for it until **after** you have differentiated.

Now think about oil being drained through a cone-shaped funnel. The rate at which the level of the oil in the funnel appears to go down can be found (as shown in the next example).

Example 4

Engine oil is added to the engine of a car. The funnel used has a radius of 10 cm and the height of the cone-shaped portion is 15 cm. (See the following diagram.) The oil is drained from the funnel at a rate of $12\ \text{cm}^3/\text{s}$. When the oil level is 5 cm from the bottom, find the rate at which the oil level is falling.



Solution

Let V be the volume of oil (in cubic centimetres) at time t, h be the height of the oil level (in centimetres), and r be the radius of the oil surface (in centimetres).

V, h, and r are variables that change as time changes.

$$V = \frac{1}{3}\pi r^2 h \qquad \text{(volume of a cone)}$$

The answer you want is $\frac{dh}{dt}$.

Since there are three variables, you must eliminate r by using a relationship between h and r. Look at the triangles in the diagram; according to the properties of similar triangles, you can say $\frac{r}{h} = \frac{10}{15}$.

Therefore, $r = \frac{2}{3}h$.

Substitute r into the equation for V.

$$V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h$$
$$= \frac{1}{3}\pi\left(\frac{4}{9}h^2\right)h$$
$$= \frac{4}{27}\pi h^3$$

Differentiate with respect to t.

$$\frac{dV}{dt} = \frac{4}{27}\pi\frac{d\left(h^3\right)}{dt}$$
$$= \frac{4}{27}\pi\left(3h^2\right)\frac{dh}{dt}$$

Substitute $\frac{dV}{dt} = -12 \text{ cm}^3/\text{s}$ and $h = 5$ into the equation.

$$-12 = \frac{4\pi}{27}(3)(5)^2\frac{dh}{dt}$$
$$\frac{dh}{dt} = -\frac{\overset{3}{\cancel{12}}(\overset{9}{\cancel{27}})}{\underset{1}{\cancel{4}}\pi(\underset{1}{\cancel{3}})(25)}$$
$$= -\frac{27}{25\pi}$$

Thus, the oil level is falling at $\frac{27}{25\pi}$ cm/s when the oil is 5 cm deep.

For additional help with related-rate problems, view the video titled *Related Rates*, from the *Catch 31* series, ACCESS Network. This video is avalaible from the Learning Resources Distributing Centre.

The following questions allow you to practice related rates for areas and volumes.

1. The area of a circle is $9\pi \text{ cm}^2$ and is increasing at the rate of $\frac{\pi}{2} \text{ cm}^2/\text{min}$. Find the rate at which the radius of the circle is increasing.

2. Coal is poured onto the top of a conical pile at the rate of $8 \text{ m}^3/\text{min}$. The diameter of the cone is three times the height. How fast is the height increasing when the pile is 4 m high?

3. A trough has ends which are isosceles trapezoids. The trough is 50 cm long, 20 cm high, 30 cm wide at the top, and 10 cm wide at the bottom. If it is filled with water at a rate of $600 \text{ cm}^3/\text{s}$, how fast is the depth increasing when the water is 10 cm deep?

In an isosceles trapezoid, the nonparallel sides are of equal length.

4. A cylindrical tank has a radius of 2 m. Water is pumped in at the rate of $3 \text{ m}^3/\text{min}$. How fast is the water level rising when the water is 4 m deep?

5. A tank, which has the shape of an inverted cone, with a height of 6 m and a radius of 5 m is being filled with water at the rate of $3\pi \text{ m}^3/\text{min}$. How fast is the water level rising when the water is 3 m deep?

6. Helium is pumped into a spherical balloon at the rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius increasing when the radius is 5 cm?

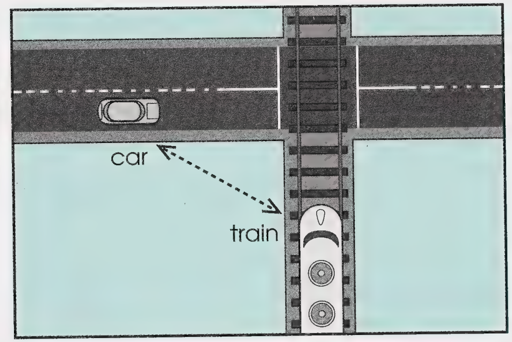
7. A trough has two ends which are equilateral triangles. The trough is 9 m long and it is 2 m wide at the top. Water is pumped in at the rate of $3 \text{ m}^3/\text{min}$. How fast does the water level rise when the deepest point is 0.5 m?

Check your answers by turning to the Appendix.

Your knowledge and appreciation of mathematics grows as your study of calculus progresses. Related rates are often expressed in subjective terms in everyday conversation.

Activity 2: Related Motion

You have probably been dealing with related rates subconsciously for a long time. If you are in a car approaching a railroad crossing at the same time that a train is approaching, what considerations do you automatically make?



The most important thing to most drivers in this situation is the rate at which the distance between the car and the train is changing. This rate of change is dependent on both the rate at which your car is travelling and the rate at which the train is travelling. It is also affected by the distances of the car and the train from the crossing.

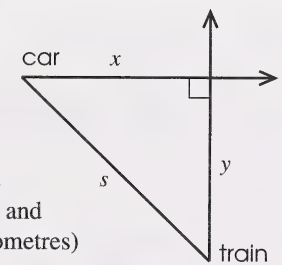
Consider a specific example of this scenario.

Example 1

A straight, level road crosses a railroad track at a right angle. A car is on the road 1 km from the crossing, travelling at 80 km/h toward the crossing. At the same time, a train, 1.2 km from the crossing, is travelling at 95 km/h toward the crossing. At what rate is the distance between the car and the train changing at that instant?

Solution

The diagram shows the relative positions of the train and the car as they approach the crossing.



Let x be the distance (in kilometres) between the car and the crossing, y be the distance (in kilometres) between the train and the crossing, and s be the distance (in kilometres) between the car and the train.

These distances are related by the Pythagorean Theorem.

$$s^2 = x^2 + y^2$$

Each of these variables is a function of time; thus, you can differentiate with respect to time.

$$\frac{d}{dt}\left(s^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right)$$
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Now you know $\frac{dx}{dt} = -80$ km/h and $\frac{dy}{dt} = -95$ km/h since both the car and the train are moving toward the crossing; thus, their distances from the crossing are decreasing.

You are also given that $x = 1$ km and $y = 1.2$ km, so you can find s using $s^2 = x^2 + y^2$.

$$\begin{aligned} s^2 &= x^2 + y^2 \\ s^2 &= 1^2 + 1.2^2 \\ s^2 &= 1 + 1.44 \\ s^2 &= 2.44 \\ s &\doteq 1.56 \end{aligned}$$

Substitute $s \doteq 1.56$, $x = 1$, and $y = 1.2$ into the differentiated equation.

$$\begin{aligned} 2(1.56)\frac{ds}{dt} &\doteq 2(1)(-80) + 2(1.2)(-95) \\ 3.12\frac{ds}{dt} &\doteq -388 \\ \frac{ds}{dt} &\doteq -124.4 \end{aligned}$$

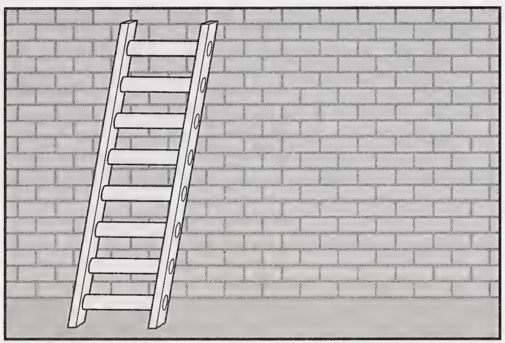
At this point the distance between the train and the car is decreasing at the rate of 124.4 km/h.

You have discovered from the previous examples that solving a related-rate problem requires that you find a relationship between the variables. Then, differentiating each of these variables implicitly with respect to time gives the relationship between the rates of change of the variables.

The derivative of a variable, like $\frac{dx}{dt}$, can be interpreted in two ways: as a rate at which a length or distance changes with time (as in the previous examples) or as the speed of a moving object. Consider this interpretation in the following examples.

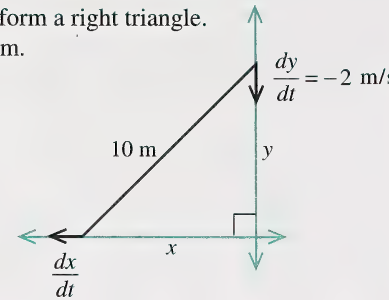
Example 6

A 10 m ladder, leaning against a wall, begins to slide down the wall. How fast is the bottom of the ladder sliding away from the base of the wall when the top of the ladder reaches 6 m up the wall and is falling at a rate of 2 m/s?



Solution

The ladder, wall, and ground form a right triangle. Use the Phythagorean Theorem.



$$x^2 + y^2 = 10^2$$

Differentiate implicitly with respect to t.

$$\frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) = \frac{d}{dt}(100)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

When $y = 6$, $x^2 + 6^2 = 10^2$

$$x^2 = 100 - 36$$

$$x^2 = 64$$

$$x = 8$$

$$\frac{dy}{dt} = -2 \text{ m/s and } x = 8$$

The -2 indicates a decreasing distance.

$$2(8)\frac{dx}{dt} + 2(6)(-2) = 0$$

$$16\frac{dx}{dt} = 24$$

$$\frac{dx}{dt} = \frac{3}{2} \text{ or } 1.5$$

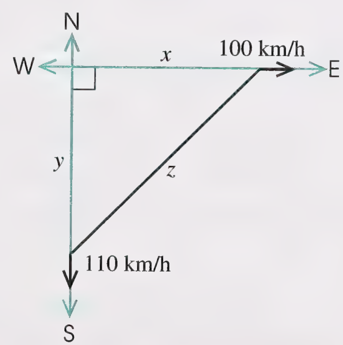
The ladder is sliding away from the base of the wall at a rate of 1.5 m/s.

Example 7

Two cars leave the city at the same time from the same location. One travels east at 100 km/h and the other travels south at 110 km/h. How fast are they separating after 1.5 h?

Solution

The positions of the two cars can be illustrated as in the diagram. The Pythagorean Theorem gives you a relationship with which to find the rate the cars are separating, $\frac{dz}{dt}$.



$$z^2 = x^2 + y^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Since you are given the speeds of the two cars, you can find their positions after 1.5 h.

$$x = 100 \times 1.5 \qquad y = 110 \times 1.5$$
$$= 150 \qquad\qquad = 165$$

Given these two values, you can find the distance between the two cars at that time.

$$z^2 = 150^2 + (165)^2$$
$$z \doteq 223$$

Substitute the values $x = 150$, $y = 165$, $z \doteq 223$, $\frac{dx}{dt} = 100$, and $\frac{dy}{dt} = 110$ to calculate $\frac{dz}{dt}$.

$$2(223)\frac{dz}{dt} \doteq 2(150)(100) + 2(165)(110)$$

$$\frac{dz}{dt} \doteq \frac{30\,000 + 36\,300}{446}$$

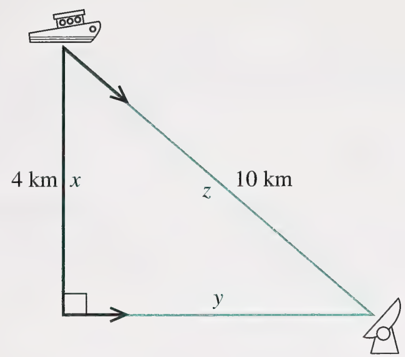
$$\doteq 148.7$$

The two cars are separating at a rate of about 148.7 km/h after 1.5 h.

Example 8

A boat is cruising parallel to the shore and is 4 km from the shoreline. A radar tracking device, situated in a lighthouse on the shore, calculates the distance between them to be decreasing at a rate of 15 km/h, when they are 10 km apart. What is the velocity of the boat?

Solution



Given $z = 10$ and $x = 4$, $y^2 = 10^2 - 4^2$

$$y = 6$$

The velocity of the boat is represented by $\frac{dy}{dt}$.

$$4^2 + y^2 = z^2$$

$$0 + 2y \bullet \frac{dy}{dt} = 2z \bullet \frac{dz}{dt}$$

There is no change in the boat's distance from shore.

$$2(6)\frac{dy}{dt} = 2(10)(-15)$$

The rate is -15 because the distance is decreasing.

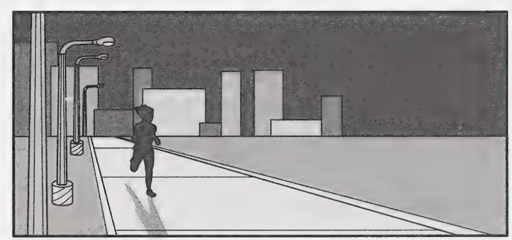
$$\frac{dy}{dt} = \frac{-300}{12} = -25$$

Velocity is negative because the boat is moving towards the radar station.

The velocity of the boat is -25 km/h. The boat's speed is 25 km/h.

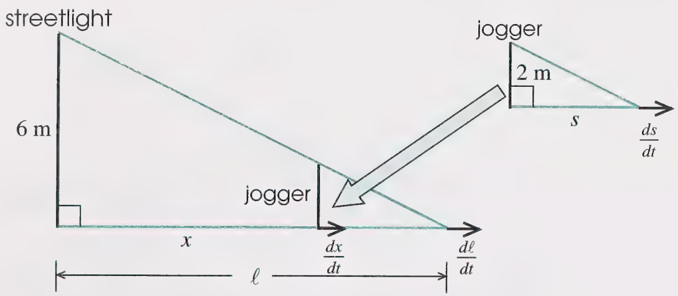
Example 9

A jogger, 2 m tall, passes a streetlight 6 m high, at a rate of 1.25 m/s. At what rate is the shadow of the jogger increasing in length? At what rate is the tip of the shadow moving?



Solution

You are asked to determine two rates. First, represent the situation with a diagram.



At any time t, s is the length of the shadow, x is the distance from the streetlight to the jogger, and ℓ is the distance from the streetlight to the tip of the shadow.

$$\ell = x + s$$

You must use the ratios of similar triangles to solve for $\frac{ds}{dt}$.

$$\begin{aligned} \frac{6}{x+s} &= \frac{2}{s} \\ 6s &= 2(x+s) \\ 6s &= 2x+2s \\ 4s &= 2x \\ s &= 0.5x \end{aligned}$$

$$\begin{aligned} \frac{ds}{dt} &= 0.5\frac{dx}{dt} \\ &= 0.51(1.25) \\ &= 0.625 \end{aligned}$$

The jogger's shadow is increasing in length at a rate of 0.625 m/s.

The speed at which the tip of the shadow is moving is found by $\frac{d\ell}{dt} = \frac{dx}{dt} + \frac{ds}{dt}$, where $\frac{d\ell}{dt}$ is the rate at which the tip of the shadow is moving, $\frac{dx}{dt}$ is the rate at which the jogger is running, and $\frac{ds}{dt}$ is the rate at which the length of the shadow is increasing.

$$\begin{aligned} \therefore \ \frac{d\ell}{dt} &= 1.25 + 0.625 \\ &= 1.875 \end{aligned}$$

The tip of the shadow is moving at a rate of 1.875 m/s.

Note that the shadow is moving faster than the man because it is growing as it moves.

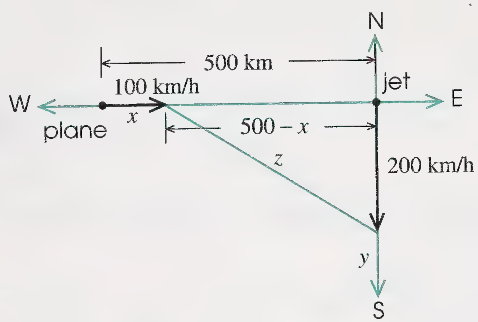
Try these questions about related motion.

1. If you are moving a 2 m post away from a light source 6 m above the ground at a rate of 1.5 m/s, at what rate is the end of the shadow of the post moving along the ground?

2. A 3 m post is leaning against a vertical wall. If the top of the post begins to slip down the wall at a rate of 0.5 m/s, how fast is the bottom of the post sliding away from the wall when it is 2 m from the base of the wall?

3. A truck leaves Edson travelling 95 km/h, and a car leaves Calgary travelling 110 km/h. Both vehicles are travelling to Edmonton. At what rate is the distance between the two vehicles decreasing when the truck is 90 km from Edmonton and the car is 120 km from Edmonton?

4. At 10:00 A.M. a plane travelling east at 100 km/h is 500 km west of a jet travelling south at 200 km/h. At what rate is the distance between them changing at noon? The following diagram illustrates the situation.



5. The curve $y = 2x^2$ is the path of a moving point. If the x-coordinate of this point is increasing at the rate of 4 units/s when $x = 3$, at what rate is the y-coordinate changing?

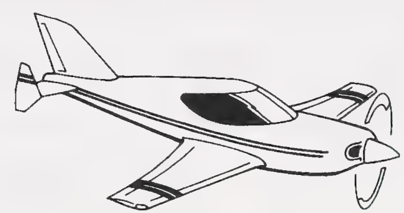
Check your answers by turning to the Appendix.

Activity 3: Using Trigonometric Functions

In the two previous activities you used algebraic concepts to solve related-rate problems. In this activity you will encounter situations that require trigonometric functions. The rules and methods of differential calculus that you used to solve the previous related-rate problems still apply, though now it will be the derivatives of trigonometric functions that will yield the solutions. You may want to look back at Module 4 to refresh your memory about the derivatives of trigonometric functions.

Example 1

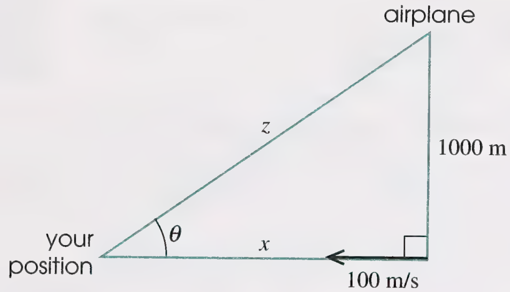
An airplane, in level flight, is approaching the spot where you are standing. The speed of the airplane is 100 m/s and it is flying at an altitude of 1000 m. What is the rate of change of the angle of elevation θ when the distance from where you are standing to a point directly below the plane is 2000 m?



Solution

Let θ be the angle of elevation.

Let x be the distance from your position to the point directly below the airplane.



You are asked to find $\frac{d\theta}{dt}$ when $x = 2000$ m. You are given that $\frac{dx}{dt} = -100$ m/s.

$$\tan\theta = \frac{1000}{x}$$

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{1000}{x}\right)$$

$$\sec^2\theta\,\frac{d\theta}{dt} = -\frac{1000}{x^2}\bullet\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1000}{x^2}\bullet\frac{dx}{dt}\bullet\cos^2\theta$$

Before you go any further, you need a value for $\cos\theta$.

When $x = 2000$, $z = \sqrt{2000^2 + 1000^2}$
$= \sqrt{5\,000\,000}$

Leaving z as an exact value makes simplifying easier later.

$$\therefore\ \cos\theta = \frac{2000}{\sqrt{5\,000\,000}}$$

$$\begin{aligned}\frac{d\theta}{dt} &= -\frac{1000}{2000^2}(-100)\left(\frac{2000}{\sqrt{5\,000\,000}}\right)^2\\ &= \left(-\frac{1000}{4\,000\,000}\right)(-100)\left(\frac{4\,000\,000}{5\,000\,000}\right)\\ &= \frac{100\,000}{5\,000\,000}\\ &= \frac{1}{50}\ \text{ or }\ 0.02\end{aligned}$$

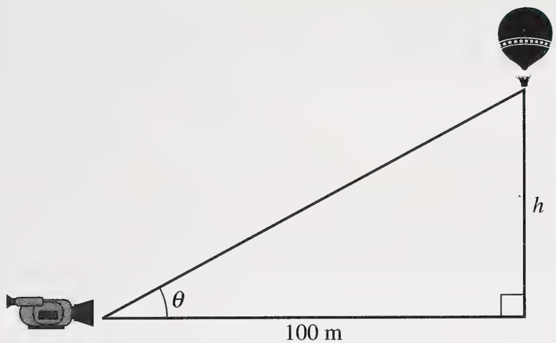
The angle of elevation is increasing at a rate of 0.02 rad/s.

Example 2

A video camera at ground level is filming the liftoff of a hot-air balloon that is rising vertically according to the position equation $h = 2t$, where h is in metres and t is in seconds. If the camera is 100 m from the launch site, find the rate of change of the angle of elevation of the camera 5 s after liftoff.

Solution

Let θ be the angle of elevation.



You are asked to find $\frac{d\theta}{dt}$, when $t = 5$.

$$\tan\theta = \frac{h}{100}$$

$$\sec^2\theta \bullet \frac{d\theta}{dt} = \frac{1}{100} \bullet \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{100} \bullet \frac{dh}{dt} \bullet \cos^2\theta$$

$$\text{When } t = 5,\ h = 2(5)$$
$$= 10$$

$$\therefore\ \cos\theta = \frac{100}{\sqrt{10^2 + 100^2}}$$
$$= \frac{100}{\sqrt{10\,100}}$$

Since $h = 2t$, $\frac{dh}{dt} = 2$.

$$\therefore\ \frac{d\theta}{dt} = \frac{1}{100} \bullet 2 \bullet \left(\frac{100}{\sqrt{10\,100}}\right)^2$$
$$= \frac{1}{50}\left(\frac{10\,000}{10\,100}\right)$$
$$= \frac{200}{10\,100}$$
$$\doteq 0.02$$

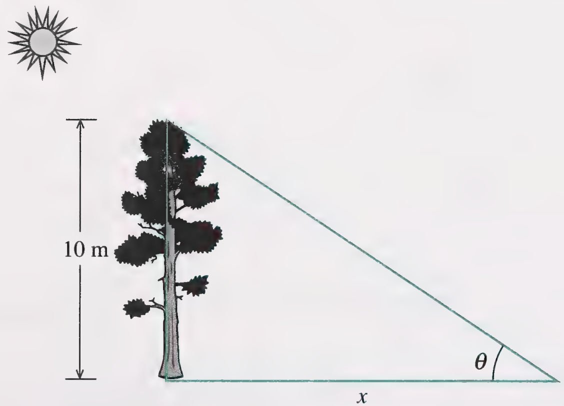
The angle of elevation is increasing at a rate of about 0.02 rad/s.

Example 3

The angle of elevation of the sun is decreasing at $\frac{1}{3}$ rad/h. How fast is the shadow cast by a tree 10 m tall lengthening when the angle of elevation of the sun is $\frac{\pi}{3}$ rad?

Solution

Let x be the length of the shadow (in metres) and θ be the angle of elevation.



You are given $\frac{d\theta}{dt} = -\frac{1}{3}$ (negative because it is decreasing).

You are asked to find the rate $\frac{dx}{dt}$ at which the shadow is lengthening.

$$\cot\theta = \frac{x}{50}$$

$$-\csc^2\theta \bullet \frac{d\theta}{dt} = \frac{1}{50} \bullet \frac{dx}{dt}$$

$$\frac{dx}{dt} = -50 \csc^2\theta \bullet \frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{3}$, $\csc\theta = 2$.

$$\therefore \frac{dx}{dt} = (-50)(2^2)\left(\frac{-1}{3}\right)$$

$$= \frac{100}{3}$$

The tree's shadow is lengthening at a rate of $\frac{100}{3}$ m/h.

These two questions involve related rates with trigonometric functions.

1. The beam of a lighthouse sweeps across the path of a boat cruising at a speed of 30 km/h parallel to the shoreline. If the boat is 2 km from the shore and stays within the beam of the light, at what rate is the beam revolving (in rad/h) when the boat has sailed 4 km from a point opposite the lighthouse.

2. A ladder 8 m long is resting against the vertical wall of a house. Fig 71 If the top of the ladder is sliding down the wall and the angle the ladder makes with the ground is decreasing at a rate of $\frac{1}{4}$ rad/s, how fast is the ladder sliding down the wall, when the angle is $\frac{\pi}{4}$ rad?

Check your answers by turning to the Appendix.

If you have attended an airshow you probably have noticed that the angle of elevation of an approaching aircraft increases more quickly as the aircraft is almost overhead.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In solving related-rate problems you again use the concept of derivative, the difference being that all the rates are differentiated with respect to time. The following steps will help you solve related-rate problems.

Step 1: Read the problem carefully.

Step 2: Represent the situation with a diagram if possible.

Step 3: Assign symbols (variables) to all quantities that are functions of time. Label your diagram accordingly.

Step 4: List all given and required rates as derivatives with respect to time.

Step 5: Write an equation that relates the quantities of the problem. This equation may reflect a well-known geometric formula for area or volume, the Pythagorean Theorem, or the properties of similar triangles. Use the geometry of the situation to eliminate one of the variables, if necessary.

Step 6: Use the chain rule to differentiate the equation with respect to time.

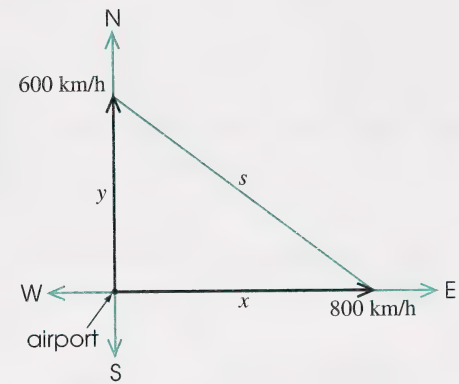
Step 7: Substitue the given information into the result of the differentiation, and solve for the unknown rate.

Note: A common error is made when the given numerical information is substituted too soon. It can only be done after differentiation.

Example

Two airplanes start out from the same airport at the same time. One flies east at 800 km/h, and the other flies north at 600 km/h. How fast are they separating after two hours?

Solution



Let the velocity of the first airplane be 800 km/h and the distance travelled by this airplane at any time be x. Let the velocity of the second airplane be 600 km/h and the distance travelled by this airplane at any time be y. Let s be the distance between the two airplanes.

After 2 h, $x = 2 \times 800 = 1600$ and $y = 2 \times 600 = 1200$.

$$\frac{dx}{dt} = 800 \text{ km/h}$$

$$\frac{dy}{dt} = 600 \text{ km/h}$$

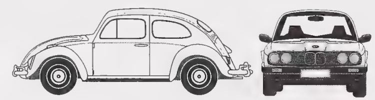
$$\begin{aligned} s^2 &= x^2 + y^2 \\ s^2 &= 1600^2 + 1200^2 \\ s^2 &= 2\,560\,000 + 1\,440\,000 \\ s^2 &= 4\,000\,000 \\ \therefore s &= 2000 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\left(s^2\right) &= \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) \\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ 2(2000)\left(\frac{ds}{dt}\right) &= 2(1600)(800) + 2(1200)(600) \\ 4000\frac{ds}{dt} &= 2\,560\,000 + 1\,440\,000 \\ \frac{ds}{dt} &= \frac{4\,000\,000}{4000} \\ &= 1000 \end{aligned}$$

Therefore, the two airplanes are separating at 1000 km/h.

Practise this concept with these two questions.

1. Two cars start out from the same place at the same time. One car travels west at 50 km/h, and the other travels south at 80 km/h. How fast are the cars separating after 2 h?



2. A rectangular container is 3 m long, 2 m wide, and 2 m deep. Water is being pumped into it at 1.5 m^3/min. How fast is the surface of the water rising?

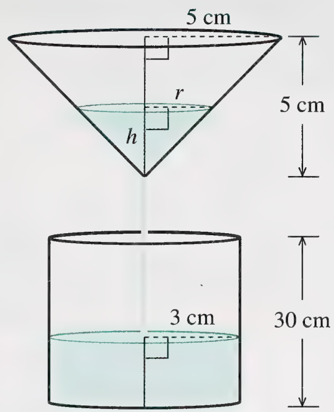
Check your answers by turning to the Appendix.

Enrichment

There are many situations in science and engineering that involve the rate of change of some property or quantity. It is impossible to have all kinds of rate problems included in this topic. The following question is different and may be more difficult.

Example

Water is being poured through a conical filter into a cylindrical container (as shown in the diagram). The filter has a radius of 5 cm and is 5 cm high. The cylinder has a radius of 3 cm and is 30 cm high. The water level in the filter is falling at $\frac{1}{2}$ cm/s when the water in the filter is 4 cm deep. How fast is the water level in the cylinder rising at that instant?



Solution

The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

$$\frac{r}{h} = \frac{5}{5}$$
$$= 1$$
$$\therefore\ r = h$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi h^2 h$$
$$= \frac{1}{3}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(3h^2\right)\frac{dh}{dt}$$

When $h = 4$ cm, $\frac{dh}{dt} = 0.5$ cm/s.

$$\therefore\ \frac{dV}{dt} = \pi(4)^2(0.5)$$
$$= 8\pi$$

This is the rate of flow out of the filter. Therefore, it must be the rate of flow into the cylinder.

The volume of the cylinder is $V = \pi r^2 H$.

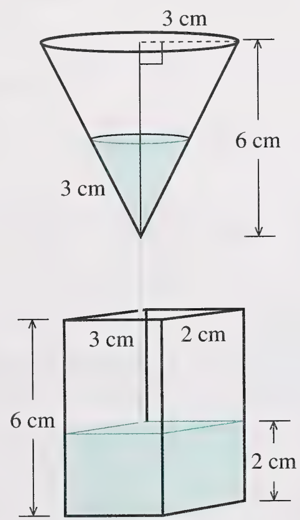
$$V = \pi r^2 H$$
$$= \pi(3)^2 H$$
$$= 9\pi H$$

$$\frac{dV}{dt} = 9\pi\frac{dH}{dt}$$
$$8\pi = 9\pi\frac{dH}{dt} \qquad \left(\text{since } \tfrac{dV}{dt} = 8\pi\right)$$
$$\frac{dH}{dt} = \frac{8\pi}{9\pi}$$
$$= \frac{8}{9}$$

The water level is rising at $\frac{8}{9}$ cm/s.

Do the following question.

Water is being poured through a conical filter into a rectangular container as shown in the diagram. The filter has a radius of 3 cm and it is 6 cm high. The rectangular container is 3 cm long, 2 cm wide, and 6 cm high. At the instant that the water level in the filter is falling at $\frac{1}{2}$ cm/s, the water in the filter is 3 cm deep, and the water level in the rectangular container is 2 cm deep. How fast is the water level in the rectangular container rising at that instant?



Check your answer by turning to the Appendix.

Conclusion

This section showed you how the derivative can be used to determine related quantities that change with respect to time. The chain rule allowed you to differentiate more than one quantity. The situations often involved well-known mathematical facts. Area and volume formulas, Pythagorean relationships, and similar triangles were used to set up an equation relating the quantities. The examples and questions showed you more applications for the concepts of calculus.

Sand draining from the top to the bottom of an hourglass marks the passage of time. Initially, the level of the sand in the upper part of the glass falls slowly, then accelerates as the top empties. Changing rates and their relationships form another important application of differential calculus.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Module Summary

In this module you developed procedures for dealing with practical problems which can be solved using derivatives. Specifically, you solved extreme-value problems by first forming the functions to be maximized or minimized; then you differentiated those functions and examined their critical values.

Maximum and minimum techniques can be used to model and explore a wide range of phenomena in the physical and biological sciences—from the shapes of crystals to the hexagonal designs of honeycombs (designs of nature conform to optimization principles).

Furthermore, you investigated the relationships among displacement, velocity, and acceleration. You also explored how rates are related, such as your walking speed to the rate at which your shadow grows.

This is your final module on differential calculus, but there is still much that you have not investigated. If you have gained an understanding of the concepts discussed to this point, you have a good chance of being successful in your further studies of calculus.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

APPENDIX

Glossary

Suggested Answers

Glossary

acceleration: the rate of change of velocity with respect to time; the derivative of the velocity function

$$a(t)=\frac{dv}{dt}$$

acceleration due to gravity: acceleration of an object in Earth's gravitational field

average acceleration: the ratio of the change in the velocity of an object to the change in time

$$a_{ave}=\frac{\text{change in velocity}}{\text{change in time}}=\frac{\Delta v}{\Delta t}=\frac{v(t_2)-v(t_1)}{t_2-t_1}$$

average cost function: the cost $C(x)$ of producing one unit of a product

$$C(x)=\frac{\text{total cost}}{\text{number of units}}=\frac{C(x)}{x}$$

average velocity: the ratio of the change in position of an object to the change in time

$$v_{ave}=\frac{\text{change in position}}{\text{change in time}}=\frac{\Delta s}{\Delta t}=\frac{s(t_2)-s(t_1)}{t_2-t_1}$$

cost function: a function $C(x)$ that determines the total cost of producing a certain number of units of a product

total cost = cost per unit × number of units

$C(x)$ = fixed costs + variable costs

displacement: the change in position of an object in a particular time interval; the interval of length between two positions

displacement $=\Delta s=s(t_2)-s(t_1)$

implicit differentiation: the technique for finding a derivative of a function in two variables without first solving for one of the variables

If $y=f(x)$ and $x=g(t)$, then differentiating with respect to time yields $\frac{dy}{dt}=\frac{dy}{dx}\bullet\frac{dx}{dt}$.

instantaneous acceleration: the acceleration function at a specified time t

$$a(t)=\frac{dv}{dt}$$

instantaneous velocity: the velocity function at a specified time t

$$v(t)=\frac{ds}{dt}$$

marginal cost: the instantaneous rate of change of cost with respect to the number of items produced

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

marginal revenue: the rate of change of revenue with respect to the number of units sold; the derivative of the revenue function

$$\text{marginal revenue} = \frac{dR}{dx}$$

optimization: the best way to perform a task, particularly when evaluating functions that are to be maximized or minimized

position function: the function $s(t)$ that describes the position of an object relative to a fixed point

revenue function: a function $R(x)$ that determines the total revenue given a price per unit $p(x)$ and the number of units sold x

$$R(x) = xp(x)$$

velocity: the rate of change of position with respect to time

$$v = \frac{\text{displacement}}{\text{time}} = \frac{ds}{dt}$$

velocity gradient: the rate of change of velocity with respect to radius of blood flow through blood vessels, assuming the vessels are a cylindrical shape

Suggested Answers

Section 1: Activity 1

1. Let x and y be the two natural numbers, and P be the product to be maximized.

$$\begin{aligned} x + y &= 28 \\ y &= 28 - x \end{aligned} \qquad \begin{aligned} P &= xy \\ &= x(28 - x) \\ &= 28x - x^2 \end{aligned}$$

The quadratic equation has a coefficient of x^2 that is negative; therefore, P has a maximum.

At a maximum, $\frac{dP}{dx} = 28 - 2x = 0$

$$\begin{aligned} 2x &= 28 \\ x &= 14 \end{aligned}$$

When $x = 14$, $y = 28 - 14$
$= 14$

Therefore, the two natural numbers are 14 and 14.

2. Let the two natural numbers be x and $(x+1)$.

$$S = \frac{4}{x} + (x+1)$$

$$\begin{aligned} \frac{dS}{dx} = -\frac{4}{x^2} + 1 &= 0 \\ 1 &= \frac{4}{x^2} \\ x^2 &= 4 \\ x &= 2 \quad (\text{Since } x \text{ is a natural number, } x \neq -2.) \end{aligned}$$

Therefore, the two numbers are 2 and $2+1=3$.

3. Let the two natural numbers be x and y.

$$\begin{aligned} x + y &= 4 \\ y &= 4 - x \end{aligned}$$

$$\begin{aligned} S &= x^3 + y^3 \\ &= x^3 + (4-x)^3 \\ &= x^3 + 64 - 48x + 12x^2 - x^3 \\ &= 64 - 48x + 12x^2 \end{aligned}$$

$$\begin{aligned} \frac{dS}{dx} = -48 + 24x &= 0 \\ 24x &= 48 \\ x &= 2 \end{aligned}$$

When $x = 2$, $y = 4 - 2$
$= 2$

The two numbers are 2 and 2.

4. Let x and y be the two numbers, and P be the product.

$$\begin{aligned} x + 2y &= 100 \\ x &= 100 - 2y \end{aligned}$$

$$\begin{aligned} P &= xy \\ &= y(100 - 2y) \\ &= 100y - 2y^2 \end{aligned}$$

$$\begin{aligned} \frac{dP}{dy} = 100 - 4y &= 0 \\ 4y &= 100 \\ y &= 25 \end{aligned}$$

$\frac{d^2P}{dy^2} = -4$; a negative implies a maximum.

When $y = 25$, $x = 100 - 2(25)$
$= 50$

The two numbers are 50 and 25.

Section 1: Activity 2

1. Let ℓ and w represent the dimensions of the rectangle.

$$\begin{aligned} A &= \ell \times w \\ 324 &= \ell \times w \\ w &= \frac{324}{\ell} \end{aligned}$$

$$\begin{aligned} P &= 2\ell + 2w \\ &= 2\ell + 2\left(\frac{324}{\ell}\right) \\ &= 2\ell + 648\ell^{-1} \end{aligned}$$

$$\begin{aligned} \frac{dP}{d\ell} = 2 - \frac{648}{\ell^2} &= 0 \\ 2 &= \frac{648}{\ell^2} \\ \ell^2 &= 324 \\ \ell &= 18 \quad (\text{positive only}) \end{aligned}$$

$$\begin{aligned} \therefore\ w &= \frac{324}{18} \\ &= 18 \end{aligned}$$

$$\frac{d^2P}{d\ell^2} = 1296\ell^{-3}$$

At $\ell = 18$, $\frac{d^2P}{d\ell^2}$ is positive.

Therefore, when $\ell = 18$ cm and $w = 18$ cm, the perimeter is a minimum.

2. Let ℓ and w represent the dimensions of the rectangle.

$$\begin{aligned} P &= 2\ell + 2w \\ 100 &= 2(\ell + w) \\ 50 &= \ell + w \\ w &= 50 - \ell \end{aligned}$$

$$\begin{aligned} A &= \ell \times w \\ &= \ell(50 - \ell) \\ &= 50\ell - \ell^2 \end{aligned}$$

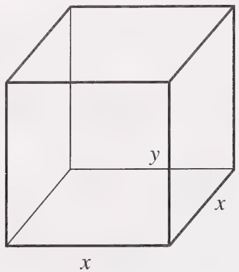
$$\begin{aligned} \frac{dA}{d\ell} = 50 - 2\ell &= 0 \\ 2\ell &= 50 \\ \ell &= 25 \end{aligned}$$

$$\begin{aligned} w &= 50 - 25 \\ &= 25 \end{aligned}$$

$$\frac{d^2A}{d\ell^2} = -2 \quad (\text{negative})$$

Therefore, the dimensions of the rectangle are 25 cm by 25 cm when the area is a maximum.

3. Let the dimensions of the box be x cm by x cm by y cm.



$$V = x^2 y$$
$$171.5 = x^2 y$$
$$y = \frac{171.5}{x^2}$$

Let M be the amount of material to be minimized.

$$\text{Surface Area} = 4xy + x^2$$
$$M = 4x\left(\frac{171.5}{x^2}\right) + x^2$$
$$= \frac{686}{x} + x^2$$

$$\frac{dM}{dx} = -\frac{686}{x^2} + 2x = 0$$
$$2x = \frac{686}{x^2}$$
$$x^3 = 343$$
$$x = 7$$

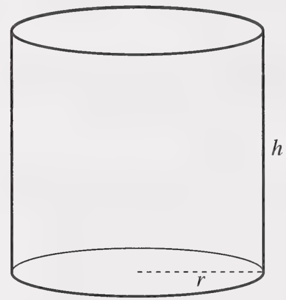
$$y = \frac{171.5}{49}$$
$$= 3.5$$

$$\frac{d^2 M}{dx^2} = \frac{1372}{x^3} + 2$$

At $x = 7$, $\frac{d^2 M}{dx^2}$ is positive; therefore, it is a minimum.

If the box is to require the least amount of material, the dimensions of the box are 7 cm by 7 cm by 3.5 cm.

4. The variables r and h are as shown in the following diagram.



$$V = \pi r^2 h$$
$$8000\pi = \pi r^2 h$$
$$h = \frac{8000}{r^2}$$

$$A = \pi r^2 + 2\pi rh$$
$$= \pi r^2 + 2\pi r\left(\frac{8000}{r^2}\right)$$
$$= \pi r^2 + \frac{16\,000\pi}{r}$$

$$\frac{dA}{dr} = 2\pi r - \frac{16\,000\pi}{r^2} = 0$$

$$2\pi r = \frac{16\,000\pi}{r^2}$$

$$r^3 = \frac{16\,000\pi}{2\pi}$$

$$r^3 = 8000$$

$$r = 20$$

When $r = 20$, $h = \frac{8000}{r^2}$

$$= \frac{8000}{400}$$

$$= 20$$

$$\frac{d^2A}{dr^2} = 2\pi + \frac{32\,000\pi}{r^3}$$

At $r = 20$, $\frac{d^2A}{dr^2}$ is positive. Therefore, if the can is to require the least amount of material, its radius should be 20 cm and its height should be 20 cm.

5. Let ℓ be the length, w be the width, and A be the area.
Let P be the perimeter to be minimized.

$$A = \ell \times w$$

$$\ell = \frac{A}{w}$$

$$P = 2w + 2\ell$$

$$= 2w + \frac{2A}{w}$$

$$\frac{dP}{dw} = 2 - \frac{2A}{w^2} = 0$$

$$2 = \frac{2A}{w^2}$$

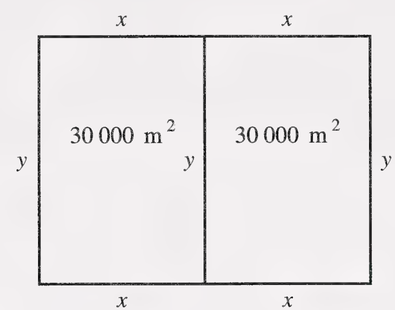
$$w^2 = A \quad (w > 0)$$

This implies a square.

$$\frac{d^2P}{dw^2} = \frac{2A}{w^3}$$

A positive second derivative implies a minimum.
A rectangle with a minimum perimeter is a square.

6.



$$A = 2xy$$

$$60\,000 = 2xy$$

$$y = \frac{60\,000}{2x}$$

$$= \frac{30\,000}{x}$$

Let P be the amount of fence.

$$\begin{aligned} P &= 3y + 4x \\ &= 3\left(\frac{30\,000}{x}\right) + 4x \\ &= \frac{90\,000}{x} + 4x \end{aligned}$$

$$\frac{dP}{dx} = -\frac{90\,000}{x^2} + 4 = 0$$

$$\begin{aligned} 4 &= \frac{90\,000}{x^2} \\ x^2 &= 22\,500 \\ x &= 150 \end{aligned}$$

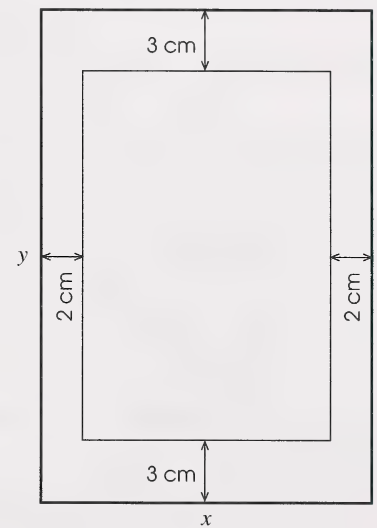
When $x = 150$, $y = \frac{30\,000}{150}$
$= 200$

$$\frac{d^2P}{dx^2} = \frac{180\,000}{x^3}$$

At $x = 150$, $\frac{d^2P}{dx^2}$ is positive.

Therefore, P is a minimum when $x = 150$ and $y = 200$.
The least amount of fence is $3(200) + 4(150) = 1200$ m.

7. Let x and y represent the dimensions of the paper.



$$\begin{aligned} \text{Total Area} &= xy \\ 600 &= xy \\ y &= \frac{600}{x} \end{aligned}$$

$$\begin{aligned} \text{Printed Area } A &= (y-6)(x-4) \\ &= \left(\frac{600}{x} - 6\right)(x-4) \\ &= 600 - \frac{2400}{x} - 6x + 24 \\ &= 624 - 6x - 2400x^{-1} \end{aligned}$$

$$\begin{aligned}\frac{dA}{dx} = -6 + \frac{2400}{x^2} &= 0\\ \frac{2400}{x^2} &= 6\\ x^2 &= 400\\ x &= 20\end{aligned}$$

When $x = 20$, $y = \frac{600}{x}$

$$\begin{aligned}&= \frac{600}{20}\\ &= 30\end{aligned}$$

$$\frac{d^2A}{dx^2} = -\frac{4800}{x^3}$$

At $x = 20$, $\frac{d^2A}{dx^2}$ is negative. Therefore, the printed area is a maximum when the dimensions of the paper are 30 cm by 20 cm.

8. Let A be the surface area to be minimized.

$$\begin{aligned}V &= x^2 y\\ 144 &= x^2 y\\ y &= \frac{144}{x^2}\end{aligned}$$

$$\begin{aligned}A &= x^2 + 3xy\\ &= x^2 + 3x\left(\frac{144}{x^2}\right)\\ &= x^2 + \frac{432}{x}\end{aligned}$$

$$\begin{aligned}\frac{dA}{dx} = 2x - \frac{432}{x^2} &= 0\\ 2x &= \frac{432}{x^2}\\ x^3 &= 216\\ x &= 6\end{aligned}$$

When $x = 6$, $y = \frac{144}{6}$

$$= 4$$

$$\frac{d^2A}{dx^2} = 2 + \frac{864}{x^3}$$

At $x = 6$, $\frac{d^2A}{dx^2}$ is positive. Therefore, the amount of netting used is a minimum when the dimensions are 6 m by 4 m by 6 m.

9. Let x be the height, $4 - 2x$ be the width, and $\frac{8-2x}{2} = 4 - x$ be the length of the box.

Let V be the volume to be maximized.

$$\begin{aligned}V &= \ell \times w \times h\\ &= (4-2x)(4-x)x\\ &= 2x^3 - 12x^2 + 16x\end{aligned}$$

$$\frac{dV}{dx} = 6x^2 - 24x + 16 = 0$$
$$3x^2 - 12x + 8 = 0$$

Use the quadratic formula to solve for *x*.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{12 \pm \sqrt{144 - 4(3)(8)}}{6} \\ &= \frac{12 \pm \sqrt{48}}{6} \\ &= \frac{12 \pm 4\sqrt{3}}{6} \\ &= 2 \pm \frac{2\sqrt{3}}{3} \end{aligned}$$

$$\therefore\ x = 2 + \frac{2\sqrt{3}}{3} \text{ or } x = 2 - \frac{2\sqrt{3}}{3}$$

When $x = 2 + \frac{2\sqrt{3}}{3}$, the volume is negative. Therefore, it is invalid.

$$\frac{d^2V}{dx^2} = 12x - 24$$

At $x = 2 - \frac{2\sqrt{3}}{3}$, $\frac{d^2V}{dx^2} = 12\left(\frac{2 - 2\sqrt{3}}{3}\right) - 24$

$$\begin{aligned} &= 24 - 8\sqrt{3} - 24 \\ &= -8\sqrt{3} \end{aligned}$$

A negative implies a maximum.

Therefore, when $x = 2 - \frac{2\sqrt{3}}{3}$ (or approximately 0.845 m), the volume will be a maximum.

10. Let *x* be the width and the height, and *y* be the length. Let *V* be the volume to be maximized.

$$x + x + y = 90$$
$$y = 90 - 2x$$

$$\begin{aligned} V &= x^2 y \\ &= x^2(90 - 2x) \\ &= 90x^2 - 2x^3 \end{aligned}$$

$$\frac{dV}{dx} = 180x - 6x^2 = 0$$
$$6x(30 - x) = 0$$
$$6x = 0 \text{ or } 30 - x = 0$$
$$x = 0 \qquad x = 30$$

Since $V = 0$ when $x = 0$, then $x \neq 0$. Therefore, $x = 30$ cm.

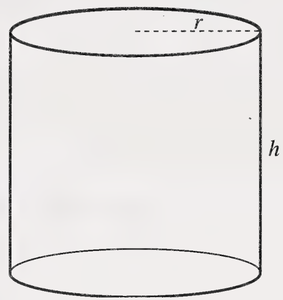
$$\begin{aligned} y &= 90 - 2(30) \\ &= 30 \end{aligned}$$

$$\frac{d^2V}{dx^2} = -12x \qquad (\text{negative})$$

The box with a maximum volume is a cube with an edge length of 30 cm.

11. $\pi r + h = 90$

$$h = 90 - \pi r$$



$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 (90 - \pi r) \\ &= 90\pi r^2 - \pi^2 r^3 \end{aligned}$$

$$\frac{dV}{dr} = 180\pi r - 3\pi^2 r^2 = 0$$

$$3\pi r(60 - \pi r) = 0$$

$3\pi r = 0$ or $60 - \pi r = 0$

$r = 0 \qquad r = \frac{60}{\pi}$

r cannot equal zero.

When $r = \frac{60}{\pi}$, $h = 90 - \pi\left(\frac{60}{\pi}\right)$

$$= 90 - 60$$
$$= 30$$

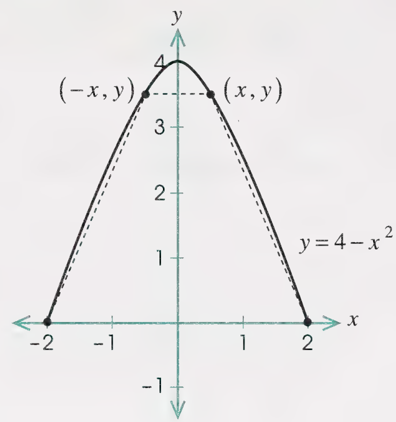
$$\frac{d^2V}{dr^2} = 180\pi - 6\pi^2 r$$

At $r = \frac{60}{\pi}$, $\frac{d^2V}{dr^2} = 180\pi - 6\pi^2\left(\frac{60}{\pi}\right)$

$$= 180\pi - 360\pi$$
$$= -180\pi \qquad (\text{maximum})$$

The cylindrical package has a maximum volume when $r = \frac{60}{\pi}$ cm and $h = 30$ cm.

12.



The x-intercepts are as follows:

$$\begin{aligned} 4 - x^2 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Therefore, the length of the longer base is $2 + 2 = 4$ units.

The height of the trapezoid is y, and the length of the shorter base is $2x$.

$$A = \frac{1}{2}y(4+2x)$$

Since $y = 4 - x^2$, $A = \frac{1}{2}(4-x^2)(4+2x)$

$$= (4-x^2)(2+x)$$
$$= 8 + 4x - 2x^2 - x^3$$

$$\frac{dA}{dx} = 4 - 4x - 3x^2 = 0$$
$$3x^2 + 4x - 4 = 0$$
$$(x+2)(3x-2) = 0$$

$x + 2 = 0$ or $3x - 2 = 0$

$x = -2$ $\quad x = \frac{2}{3}$

If $x = -2$, the area $= 0$; thus, x cannot be -2.

When $x = \frac{2}{3}$, $y = 4 - \left(\frac{2}{3}\right)^2$

$$= 4 - \frac{4}{9}$$
$$= \frac{32}{9} \text{ or } 3\frac{5}{9}$$

$$A = \frac{1}{2}\left[4 + \left(2 \times \frac{2}{3}\right)\right] \times \frac{32}{9}$$
$$= \frac{1}{\cancel{2}_1}\left(\frac{16}{3}\right)\left(\frac{\cancel{32}^{16}}{9}\right)$$
$$= \frac{256}{27} \text{ or } 9\frac{13}{27}$$

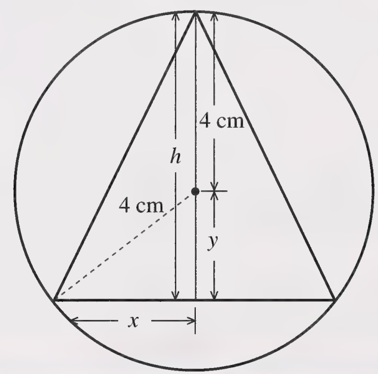
$$\frac{d^2A}{dx^2} = -6x - 4$$

At $x = \frac{2}{3}$, $\frac{d^2A}{dx^2} = -6\left(\frac{2}{3}\right) - 4$

$= -8$ (negative)

Therefore, $9\frac{13}{27}$ is a maximum area.

13. Represent the dimensions of the isosceles triangle as shown.



$$x^2 + y^2 = 4^2, \text{ where } y = h-4$$
$$x^2 = 16 - y^2$$
$$x^2 = 16 - (h-4)^2$$
$$x^2 = 16 - \left(h^2 - 8h + 16\right)$$
$$x^2 = -h^2 + 8h$$
$$x = \sqrt{8h - h^2}$$

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(2x)(h)$$
$$= xh$$
$$= \sqrt{8h-h^2} \bullet h$$

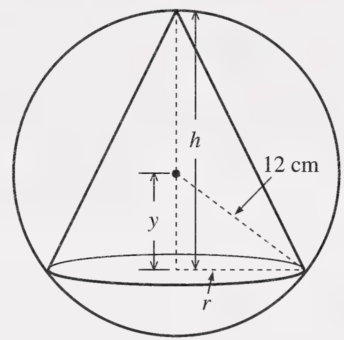
$$\frac{dA}{dh} = h \bullet \frac{1}{2}\left(8h-h^2\right)^{-\frac{1}{2}}(8-2h) + \left(8h-h^2\right)^{\frac{1}{2}} = 0$$
$$\frac{8h-2h^2}{2\left(8h-h^2\right)^{\frac{1}{2}}} + \frac{\left(8h-h^2\right)^{\frac{1}{2}}}{1} = 0$$
$$\frac{8h-2h^2+2\left(8h-h^2\right)}{2\left(8h-h^2\right)^{\frac{1}{2}}} = 0$$
$$8h-2h^2+16h-2h^2 = 0$$
$$-4h^2+24h = 0$$
$$-4h(h-6) = 0$$

$$-4h = 0 \quad \text{or} \quad h-6=0$$
$$h=0 \qquad h=6 \qquad (\text{Discard } h=0.)$$

When $h=6$, $x = \sqrt{8\times6-6^2}$
$$= \sqrt{12}$$
$$= 2\sqrt{3}$$

The height of the triangle is 6 cm and the base is $2\left(4\sqrt{3}\right) = 4\sqrt{3}$ cm when the area is a maximum.

14.



$$y^2 + r^2 = 12^2, \text{ where } y = h-12$$
$$r^2 = 12^2 - y^2$$
$$r^2 = 144 - (h-12)^2$$
$$r^2 = 144 - h^2 + 24h - 144$$
$$r^2 = 24h - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(24h-h^2\right)h$$
$$= \frac{\pi}{3}\left(24h^2 - h^3\right)$$

$$\frac{dV}{dh} = \frac{\pi}{3}\left(48h - 3h^2\right) = 0$$
$$48h - 3h^2 = 0$$
$$h(48 - 3h) = 0$$

$$h = 0 \text{ or } 48 - 3h = 0$$
$$3h = 48$$
$$h = 16$$

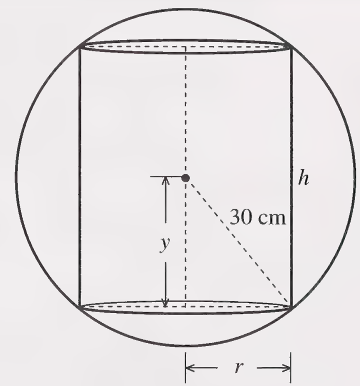
$h \neq 0$

$$r^2 = 24(16) - 16^2$$
$$r^2 = 128$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(128) \times 16$$
$$= \frac{2048}{3}\pi$$

The maximum volume is $\frac{2048}{3}\pi$ cm^3 or $682\frac{2}{3}\pi$ cm^3.

15. The dimensions of the cylinder are as shown in the diagram.



$$y^2 + r^2 = 30^2$$
$$r^2 = 30^2 - y^2$$

Since $y = \frac{h}{2}$, $r^2 = 30^2 - \left(\frac{h}{2}\right)^2$
$$= 900 - \frac{h^2}{4}$$

$$V = \pi r^2 h$$
$$= \pi\left(900 - \frac{h^2}{4}\right)h$$
$$= \pi\left(900h - \frac{h^3}{4}\right)$$

$$\frac{dV}{dh} = \pi\left(900 - \frac{3h^2}{4}\right) = 0$$

$$900 - \frac{3h^2}{4} = 0$$

$$\frac{3h^2}{4} = 900$$

$$h^2 = 1200$$

$$h = 20\sqrt{3}$$

At $h = 20\sqrt{3}$, $r^2 = 900 - \frac{h^2}{4}$

$$r = \sqrt{900 - \frac{\left(20\sqrt{3}\right)^2}{4}}$$

$$= \sqrt{900 - 300}$$

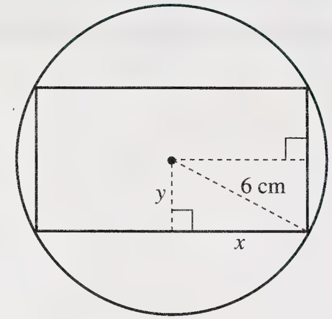
$$= \sqrt{600}$$

$$= 10\sqrt{6}$$

Therefore, the radius of the right circular cylinder is $10\sqrt{6}$ cm and the height is $20\sqrt{3}$ cm.

16. $x^2 + y^2 = 6^2$

$$y = \sqrt{36 - x^2}$$



$$A = (2x)(2y)$$

$$= 4xy$$

$$= 4x\left(36 - x^2\right)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = 4x\left(\frac{1}{2}\right)\left(36 - x^2\right)^{-\frac{1}{2}}(-2x) + \left(36 - x^2\right)^{\frac{1}{2}}(4) = 0$$

$$-4x^2\left(36 - x^2\right)^{-\frac{1}{2}} + 4\left(36 - x^2\right)^{\frac{1}{2}} = 0$$

$$\frac{-4x^2 + 4\left(36 - x^2\right)}{\left(36 - x^2\right)^{\frac{1}{2}}} = 0$$

$$-8x^2 + 144 = 0$$

$$8x^2 = 144$$

$$x^2 = 18$$

$$x = 3\sqrt{2}$$

At $x = 3\sqrt{2}$, $y = \sqrt{36 - 18}$

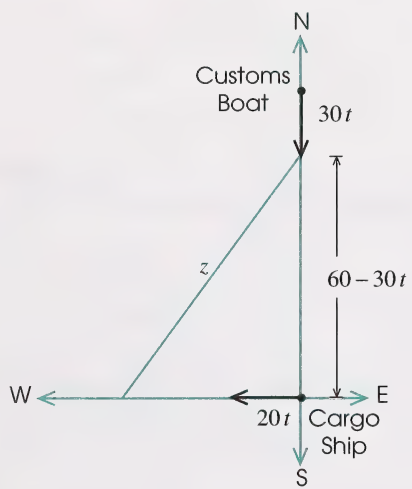
$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

The dimensions of the rectangle are $2x$ by $2y$; thus, its dimensions are $6\sqrt{2}$ cm by $6\sqrt{2}$ cm.

Section 1: Activity 3

1. Let t be the time travelled.
   Let z be the distance to be minimized.



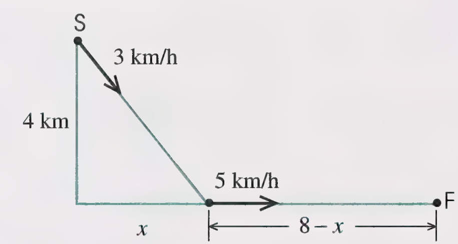
$$z^2 = (60-30t)^2 + (20t)^2$$
$$z^2 = 3600 - 3600t + 900t^2 + 400t^2$$
$$z^2 = 1300t^2 - 3600t + 3600$$

$$2z \bullet \frac{dz}{dt} = 2600t - 3600$$
$$\frac{dz}{dt} = \frac{1300t-1800}{z} = 0$$
$$1300t - 1800 = 0$$
$$t = \frac{1800}{1300} \qquad \boxed{\frac{d^2z}{dt^2} = 1300 \quad (\text{minimum})}$$
$$\doteq 1.38$$
$$\doteq 1 \text{ h } 23 \text{ min}$$

Therefore, the boats are closest together at about 11:23 A.M.

2. Let x be the distance on the shoreline from the point directly across from the launch to the point where the swimmer begins to walk.

   Let T be the total time to be to be minimized.



$$T = \text{swimming time} + \text{walking time}$$
$$= \frac{8-x}{5} + \frac{\sqrt{x^2+16}}{3}$$
$$= \frac{1}{5}(8-x) + \frac{1}{3}\left(x^2+16\right)^{\frac{1}{2}}$$

$$\frac{dT}{dx} = -\frac{1}{5} + \frac{1}{3}\left(\frac{1}{2}\right)\left(x^2+16\right)^{-\frac{1}{2}}(2x) = 0$$

$$-\frac{1}{5} + \frac{x}{3\sqrt{x^2+16}} = 0$$

$$\frac{x}{3\sqrt{x^2+16}} = \frac{1}{5}$$

$$5x = 3\sqrt{x^2+16}$$

$$25x^2 = 9x^2+144$$

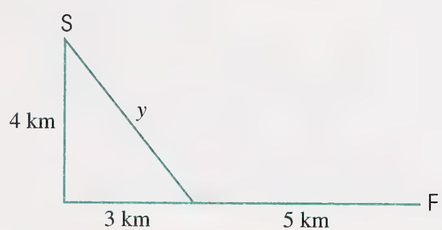
$$16x^2 = 144$$

$$x^2 = 9$$

$$x = 3$$

Therefore, to minimize the length of the race, the swimmer should aim 3 km down the shore from a point directly across from the launch.

3. Let y be the distance swam.



$$y^2 = 3^2 + 4^2$$

$$y^2 = 25$$

$$y = 5$$

$$\therefore \text{ Total Distance} = 5+5 = 10$$

The minimum distance of the race is 10 km.

Section 1: Activity 4

1. $C = 250 + \dfrac{h}{10} + \dfrac{2\,500\,000}{h}$

$$\frac{dC}{dh} = \frac{1}{10} + \left(-\frac{2\,500\,000}{h^2}\right) = 0$$

$$\frac{1}{10} = \frac{2\,500\,000}{h^2}$$

$$h^2 = 25\,000\,000$$

$$h = 5000$$

The height is 5000 m.

2. Assume that the rent is increased by $25x$, where $x \in N$. Rent per unit is equal to $460+25x$. The number of units occupied is equal to $150-4x$.

$$\begin{aligned}\text{Revenue} &= (460+25x)(150-4x)\\ &= 69\,000 + 3750x - 1840x - 100x^2\\ &= -100x^2 + 1910x + 69\,000\end{aligned}$$

$$\begin{aligned}\text{Total Cost} &= (150 - 4x) \times 72.50 \\ &= 10\,875 - 290x\end{aligned}$$

$$\begin{aligned}\text{Profit } (P) &= \text{Revenue} - \text{Cost} \\ &= -100x^2 + 1910x + 69\,000 - (-290x + 10\,875) \\ &= -100x^2 + 1910x + 69\,000 + 290x - 10\,875 \\ &= -100x^2 + 2200x + 58\,125\end{aligned}$$

At a maximum, $\frac{dP}{dx} = 0$.

$$\begin{aligned}\frac{dP}{dx} = -200x + 2200 &= 0 \\ 200x &= 2200 \\ x &= 11\end{aligned}$$

$$\begin{aligned}\text{Rent charged} &= \$460 + \$25(11) \\ &= \$735\end{aligned}$$

In order to realize the most profit, the rent charged should be \$735.

3. Let x be the number of rooms occupied and $(40 - x)$ be the number of vacancies.

Let R be the revenue to be maximized.

$$\begin{aligned}p(x) &= \text{price per unit} \\ &= 60 + 5(40 - x) \\ &= 260 - 5x\end{aligned}$$

$$\begin{aligned}R(x) &= xp(x) \\ &= x(260 - 5x) \\ &= 260x - 5x^2\end{aligned}$$

$$\begin{aligned}\frac{dR}{dx} = 260 - 10x &= 0 \\ 10x &= 260 \\ x &= 26\end{aligned}$$

$$\begin{aligned}p(26) &= 260 - 5(26) \\ &= 130\end{aligned}$$

$$\begin{aligned}R(x) &= 26(130) \\ &= 3380\end{aligned}$$

The rooms should be priced at \$130 per night to earn a maximum revenue of \$3380.

4. Let x be the number of customers and $(15-x)$ be the number of customers lost.

Let P be the profit to be maximized.

$$\begin{aligned} p(x) &= \text{price per job} \\ &= 15+2(15-x) \\ &= 45-2x \end{aligned}$$

$$\begin{aligned} R(x) &= \text{Revenue} \\ &= x(45-2x) \\ &= 45x-2x^2 \end{aligned}$$

$$\begin{aligned} C(x) &= \text{Cost} \\ &= xc(x) \\ &= 5x \end{aligned}$$

$$\frac{dR}{dx}=45-4x$$

$$\frac{dC}{dx}=5$$

To maximize profit, $\frac{dR}{dx}=\frac{dC}{dx}$.

$$\begin{aligned} \frac{dR}{dx} &= \frac{dC}{dx} \\ 45-4x &= 5 \\ 4x &= 40 \\ x &= 10 \end{aligned}$$

$\frac{d^2R}{dx^2}=-4$; therefore, $x=10$ yields a maximum.

$$\begin{aligned} \therefore p(x) &= 45-2(10) \\ &= 25 \end{aligned}$$

$$\begin{aligned} \therefore R(x) &= 10(25) \\ &= 250 \end{aligned}$$

$$\begin{aligned} \therefore C(x) &= 10(5) \\ &= 50 \end{aligned}$$

$$\begin{aligned} \therefore P(x) &= R(x)-C(x) \\ &= 250-50 \\ &= 200 \end{aligned}$$

If Dana charges \$25 per job, she will have 10 customers and make \$200 profit.

5.
$$E=\frac{\left(m-\frac{1}{3}m^2\right)}{m+\frac{1}{3}}$$

$$\begin{aligned} \frac{dE}{dm} &= \frac{\left(m+\frac{1}{3}\right)D_m\left(m-\frac{1}{3}m^2\right)-\left(m-\frac{1}{3}m^2\right)D_m\left(m+\frac{1}{3}\right)}{\left(m+\frac{1}{3}\right)^2} \\ &= \frac{\left(m+\frac{1}{3}\right)\left(1-\frac{2}{3}m\right)-\left(m-\frac{1}{3}m^2\right)}{\left(m+\frac{1}{3}\right)^2} \end{aligned}$$

At a maximum, $\frac{dE}{dm} = 0$.

$$\frac{\left(m+\frac{1}{3}\right)\left(1-\frac{2}{3}m\right)-\left(m-\frac{1}{3}m^2\right)}{\left(m+\frac{1}{3}\right)^2} = 0$$

$$\left(m+\frac{1}{3}\right)\left(1-\frac{2}{3}m\right)-\left(m-\frac{1}{3}m^2\right) = 0$$

$$m-\frac{2}{3}m^2+\frac{1}{3}-\frac{2}{9}m-m+\frac{1}{3}m^2 = 0$$

$$-\frac{1}{3}m^2-\frac{2}{9}m+\frac{1}{3} = 0$$

$$3m^2+2m-3 = 0$$

Use the quadratic formula.

$$m = \frac{-2\pm\sqrt{2^2-4(3)(-3)}}{2\times 3}$$

$$= \frac{-2\pm\sqrt{40}}{6}$$

$$= \frac{-2\pm 2\sqrt{10}}{6}$$

$$= \frac{-1\pm\sqrt{10}}{3}$$

Since the tangent of the angle of pitch is always positive, $m = \frac{-1+\sqrt{10}}{3}$ is the value of m when E is a maximum. The pitch angle is about 35.8°.

6. $I = \frac{E}{R+r}$

$$P = I^2 R$$

$$= \left(\frac{E}{R+r}\right)^2 R$$

$$= E^2 (R+r)^{-2} R$$

$$\frac{dP}{dR} = E^2\left[RD_R(R+r)^{-2}+(R+r)^{-2}D_R R\right]$$

$$= E^2\left[R(-2)(R+r)^{-3}+(R+r)^{-2}\right]$$

$$= E^2\left[\frac{-2R}{(R+r)^3}+\frac{1}{(R+r)^2}\right]$$

$$E^2 = \left[\frac{-2R+(R+r)}{(R+r)^3}\right]$$

At a maximum, $\frac{dP}{dR} = 0$.

$$-2R+(R+r) = 0$$

$$-R+r = 0$$

$$R = r$$

If $r = 0.2\ \Omega$, then $R = 0.2\ \Omega$.

The external resistance is $0.2\ \Omega$ when the power is a maximum.

7. If $P = N + \frac{Nt}{t^2 + 64}$, then $P = 2000 + \frac{2000t}{t^2 + 64}$.

$$\frac{dP}{dt} = 0 + \frac{\left(t^2 + 64\right)D_t\left(2000t\right) - 2000tD_t\left(t^2 + 64\right)}{\left(t^2 + 64\right)^2}$$
$$= \frac{\left(t^2 + 64\right)2000 - 2000t\left(2t\right)}{\left(t^2 + 64\right)^2}$$
$$= \frac{2000t^2 + 128\,000 - 4000t^2}{\left(t^2 + 64\right)^2}$$
$$= \frac{-2000t^2 + 128\,000}{\left(t^2 + 64\right)^2}$$

At a maximum, $\frac{dP}{dt} = 0$.

$$-2000t^2 + 128\,000 = 0$$
$$2000t^2 = 128\,000$$
$$t^2 = 64$$
$$t = 8$$

Time cannot be negative; therefore, $t = 8$.

When $t = 8$, $P = 2000 + \frac{2000(8)}{8^2 + 64}$

$$= 2000 + \frac{16\,000}{128}$$
$$= 2125$$

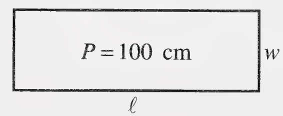
The maximum size of this population is 2125 when $t = 8$ h.

Section 1: Follow-up Activities

Extra Help

1. **Step 1:** Read the problem carefully.

Step 2: Draw a diagram.



Step 3: Let ℓ be the length of the rectangle and w be the width of the rectangle.

Let P be the perimeter of the rectangle and A be the area.

Step 4:

$$P = 2\ell + 2w$$
$$100 = 2(\ell + w)$$
$$50 = \ell + w$$
$$w = 50 - \ell$$

$$A = \ell \times w$$
$$= \ell(50 - \ell)$$
$$= 50\ell - \ell^2$$

Step 5: $\frac{dA}{d\ell} = 50 - 2\ell$

At a maximum, $\frac{dA}{d\ell} = 0$.

$$50 - 2\ell = 0$$
$$2\ell = 50$$
$$\ell = 25$$

Step 6: Determine the width of the rectangle.

$$\begin{aligned} w &= 50 - \ell \\ &= 50 - 25 \\ &= 25 \end{aligned}$$

Step 7: Verify that your result is a maximum.

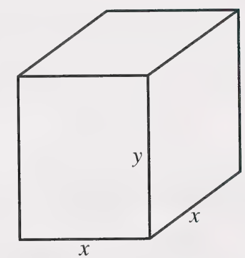
$$\frac{D^2 A}{d\ell^2} = -2 \quad \text{(negative)}$$

Thus, the area is a maximum.

Step 8: The dimensions of the rectangle are 25 cm by 25 cm.

2. **Step 1:** Read the problem carefully.

Step 2: Draw a diagram.



Step 3: Let the dimensions of the base be x by x.
Let y be the height of the box.
Let A be the total surface area and V be the volume.

Step 4: $A = x^2 + 4xy = 48$

$$48 = x^2 + 4xy$$
$$y = \frac{48 - x^2}{4x}$$

$$\begin{aligned} V &= x^2 y \\ &= x^2\left(\frac{48 - x^2}{4x}\right) \\ &= \frac{48x - x^3}{4} \end{aligned}$$

Step 5: At a maximum, $\frac{dV}{dx}=0$.

$$\begin{aligned}\frac{dV}{dx}=12-\frac{3}{4}x^2 &= 0\\ \frac{3}{4}x^2 &= 12\\ x^2 &= 16\\ x &= 4\end{aligned}$$

Step 6: Solve for your unknown.

$$\begin{aligned}\text{At } x=4,\ y&=\frac{48-4^2}{4(4)}\\ &=\frac{48-16}{16}\\ &=2\end{aligned}$$

Step 7: Verify that your result is a maximum.

$$\frac{d^2V}{dx^2}=-\frac{3}{2}x$$

At $x=4,\ \frac{d^2V}{dx^2}=-6.$

Since the second derivative is negative, it represents a maximum volume.

Step 8: The dimensions of the box are 4 cm by 4 cm by 2 cm and the maximum volume is $32\ \text{cm}^3$.

Enrichment

1. $y=x^4+4x^3-8x^2-48x+4$

$$\begin{aligned}\frac{dy}{dx}=4x^3+12x^2-16x-48&=0\\ x^3+3x^2-4x-12&=0\end{aligned}$$

Factoring gives you the following:

$$\begin{aligned}x^2(x+3)-4(x+3)&=0\\ \left(x^2-4\right)(x+3)&=0\\ (x+3)(x+2)(x-2)&=0\end{aligned}$$

$x+3=0$ or $x+2=0$ or $x-2=0$

$x=-3$ $\quad x=-2$ $\quad x=2$

$$\frac{d^2y}{dx^2}=12x^2+24x-16$$

When $x=-3,\ y=(-3)^4+4(-3)^3-8(-3)^2-48(-3)+4$

$$\begin{aligned}&=81-108-72+144+4\\ &=49\end{aligned}$$

$$\frac{d^2y}{dx^2} = 12(-3)^2 + 24(-3) - 16$$
$$= 108 - 72 - 16$$
$$= 20 \quad (\text{positive})$$

Thus, 49 is a minimum when $x = -3$.

When $x = -2$, $y = (-2)^4 + 4(-2)^3 - 8(-2)^2 - 48(-2) + 4$
$$= 16 - 32 - 32 + 96 + 4$$
$$= 52$$

$$\frac{d^2y}{dx^2} = 12(-2)^2 + 24(-2) - 16$$
$$= 48 - 48 - 16$$
$$= -16 \quad (\text{negative})$$

Thus, 52 is a maximum when $x = -2$.

When $x = 2$, $y = (2)^4 + 4(2)^3 - 8(2)^2 - 48(2) + 4$
$$= 16 + 32 - 32 - 96 + 4$$
$$= -76$$

$$\frac{d^2y}{dx^2} = 12(2)^2 + 24(2) - 16$$
$$= 48 + 48 - 16$$
$$= 80 \quad (\text{positive})$$

Thus, -76 is a minimum when $x = 2$.

2. $y = \frac{x^4}{2} + x^3 - 4x^2 + 3x - 1$

$$\frac{dy}{dx} = 2x^3 + 3x^2 - 8x + 3 = 0$$

Factor using synthetic division.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & \boxed{0} \end{array}$$

$$\therefore \ (x-1)\left(2x^2 + 5x - 3\right) = 0$$
$$(x-1)(2x-1)(x+3) = 0$$

$x + 3 = 0$ or $2x - 1 = 0$ or $x - 1 = 0$

$x = -3$ $\qquad x = \frac{1}{2}$ $\qquad x = 1$

$$\frac{d^2y}{dx^2} = 6x^2 + 6x - 8$$

When $x=-3$, $y=\dfrac{(-3)^4}{2}+(-3)^3-4(-3)^2+3(-3)-1$

$$=\frac{81}{2}-27-36-9-1$$
$$=-32.5$$

$$\frac{d^2y}{dx^2}=6(-3)^2+6(-3)-8$$
$$=54-18-8$$
$$=28 \quad (\text{positive})$$

Thus, -32.5 is a minimum when $x=-3$.

When $x=\frac{1}{2}$, $y=\dfrac{\left(\frac{1}{2}\right)^4}{2}+\left(\frac{1}{2}\right)^3-4\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)-1$

$$=\frac{1}{32}+\frac{1}{8}-1+\frac{3}{2}-1$$
$$=\frac{1+4-32+48-32}{32}$$
$$=-\frac{11}{32}$$

$$\frac{d^2y}{dx^2}=6\left(\frac{1}{2}\right)^2+6\left(\frac{1}{2}\right)-8$$
$$=\frac{3}{2}+3-8$$
$$=-3.5 \quad (\text{negative})$$

Thus, $-\frac{11}{32}$ is a maximum when $x=\frac{1}{2}$.

When $x=1$, $y=\dfrac{(1)^4}{2}+(1)^3-4(1)^2+3(1)-1$

$$=\frac{1}{2}+1-4+3-1$$
$$=-\frac{1}{2}$$

$$\frac{d^2y}{dx^2}=6(1)^2+6(1)-8$$
$$=6+6-8$$
$$=4 \quad (\text{positive})$$

Thus, $-\frac{1}{2}$ is a minimum when $x=1$.

Section 2: Activity 1

1. a. The displacement of the car is 150 km.

b. $v_{ave}=\dfrac{\text{displacement}}{\text{time}}$

$$=\frac{150}{1.5+2.5+3}$$
$$=\frac{150}{7} \text{ or } 21\frac{3}{7}$$

The average velocity is $21\frac{3}{7}$ km/h.

c. $$\text{Average speed} = \frac{\text{distance}}{\text{time}} = \frac{150+250+250}{1.5+2.5+3} = \frac{650}{7} \text{ or } 92\frac{6}{7}$$

The average speed is $92\frac{6}{7}$ km/h.

2. a. $$\text{Displacement} = (90\times 3)+(100\times 4) = 270+400 = 670$$

The displacement is 670 km.

b. $$v_{ave} = \frac{670}{7} = 95\frac{5}{7}$$

The average velocity is $95\frac{5}{7}$ km/h.

3. a. $$h = -5t^2 + 45t + 25$$

$$v = \frac{dh}{dt} = -10t + 45$$

b. When $t = 1$, $v = -10(1) + 45 = 35$

The velocity at $t = 1$ is 35 m/s.

c. When $v = 0$, the object reaches its maximum displacement.

$$-10t + 45 = 0$$
$$10t = 45$$
$$t = 4.5$$

The object reaches its maximum displacement when $t = 4.5$ s.

d. When $t = 4.5$, $h = -5(4.5)^2 + 45(4.5) + 25 = -101.25 + 202.5 + 25 = 126.25$

The maximum displacement is 126.25 m.

e. When $t = 2$, $h = -5(2)^2 + 45(2) + 25 = -20 + 90 + 25 = 95$

When $t = 0$, $h = -5(0)^2 + 45(0) + 25 = 0 + 0 + 25 = 25$

$$v_{ave} = \frac{95-25}{2-0} = \frac{70}{2} = 35$$

The average velocity from $t=0$ to $t=2$ is 35 m/s.

4. a. $$v_{ave} = \frac{s(4)-s(1)}{4-1} = \frac{\left[4^3-9(4)^2+24(4)\right]-\left[1^3-9(1)^2+24(1)\right]}{3} = \frac{16-16}{3} = 0$$

Since $v_{ave}=0$, there is no change in the **average** velocity in this time interval.

b. $v(t)=3t^2-18t+24$

c. $$v(3)=3(3)^2-18(3)+24 = -3 \text{ m/s}$$

d. The minimum of s is found when $\frac{ds}{dt}=v=0$.

$$3t^2-18t+24=0$$
$$t^2-6t+8=0$$
$$(t-2)(t-4)=0$$
$$t-2=0 \text{ or } t-4=0$$
$$t=2 \qquad t=4$$

Now $\frac{d^2s}{dt^2}=6t-18$.

When $t=2$, $6t-18=6(2)-18 = -6$

A negative yields a maximum.

When $t=4$, $6t-18=6(4)-18 = 6$

A positive yields a minimum.

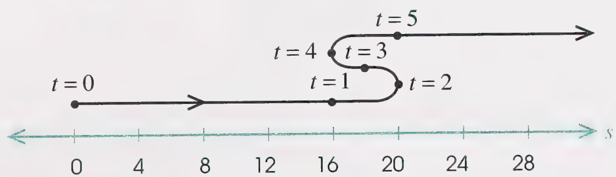
Therefore, the minimum position occurs at $t=4$.

e. $$s(2)=(2)^3-9(2)^2+24(2) = 8-36+48 = 20 \text{ m}$$

$$s(4)=(4)^3-9(4)^2+24(4) = 64-144+96 = 16 \text{ m}$$

f.

t	0	1	2	3	4	5
$s(t)$	0	16	20	18	16	20



Section 2: Activity 2

1. a. $a = \dfrac{dv}{dt} = 2 - 6t$

When $t = 4$, $a = 2 - 6(4)$

$= 2 - 24$

$= -22$

b. $a = \dfrac{dv}{dt} = 2(t+3)$

When $t = 3$, $a = 2(3+3)$

$= 12$

c. $a = \dfrac{dv}{dt} = 15t^2 - 2$

When $t = 2$, $a = 15(2)^2 - 2$

$= 60 - 2$

$= 58$

d. $a = \dfrac{dv}{dt} = \dfrac{(3+t^2)\frac{d}{dt}(10t) - (10t)\frac{d}{dt}(3+t^2)}{(3+t^2)^2}$

$= \dfrac{(3+t^2)(10) - (10t)(2t)}{(3+t^2)^2}$

$= \dfrac{30 + 10t^2 - 20t^2}{(3+t^2)^2}$

$= \dfrac{-10t^2 + 30}{(3+t^2)^2}$

When $t = 1$, $a = \dfrac{-10(1)^2 + 30}{(3+1^2)^2}$

$= \dfrac{20}{16}$

$= 1\dfrac{1}{4}$

2. a. $v = \dfrac{ds}{dt} = 2t - 5$

$a = \dfrac{d^2s}{dt^2} = 2$

b. $2t-5=0$

$$t=\frac{5}{2} \text{ or } 2\frac{1}{2}$$

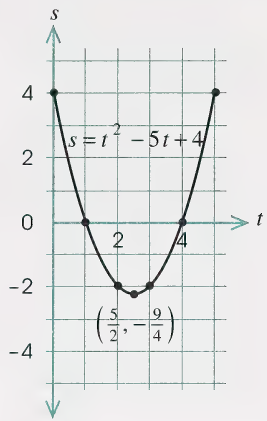
When $t=\frac{5}{2}$, $s=\left(\frac{5}{2}\right)^2-5\left(\frac{5}{2}\right)+4$

$$=-2\frac{1}{4}$$

Therefore, the turning point is $\left(\frac{5}{2},-2\frac{1}{4}\right)$.

c.

t	s
0	4
1	0
2	−2
$2\frac{1}{2}$	$-2\frac{1}{4}$
3	−2
4	0
5	4



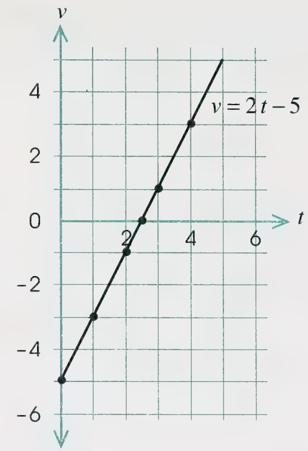
d. When $t=1$, the direction of the motion is negative.

When $t=\frac{5}{2}$, the object stops.

When $t=3$, the direction of the motion is positive.

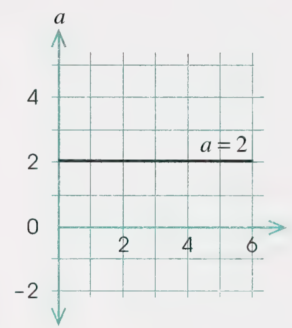
e.

t	v
0	−5
1	−3
2	−1
$\frac{5}{2}$	0
3	1
4	3



The velocity is increasing at a constant rate.

f. $a=\frac{dv}{dt}=2$



The acceleration is a constant 2.

3. **a.** $s = 2t^3 - 15t^2 + 24t + 8,\ t \geq 0$

$$v = \frac{ds}{dt} = 6t^2 - 30t + 24$$

$$a = \frac{dv}{dt} = 12t - 30$$

b. $6t^2 - 30t + 24 = 0$

$$t^2 - 5t + 4 = 0$$

$$(t-4)(t-1) = 0$$

$$t - 4 = 0 \text{ or } t - 1 = 0$$

$$t = 4 \qquad t = 1$$

When $t = 4$, $s = 2(4)^3 - 15(4)^2 + 24(4) + 8$

$$= 128 - 240 + 96 + 8$$

$$= -8$$

When $t = 1$, $s = 2(1)^3 - 15(1)^2 + 24(1) + 8$

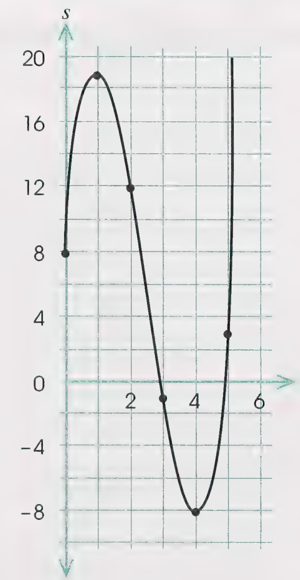
$$= 2 - 15 + 24 + 8$$

$$= 19$$

The two turning points are $(4, -8)$ and $(1, 19)$.

c.

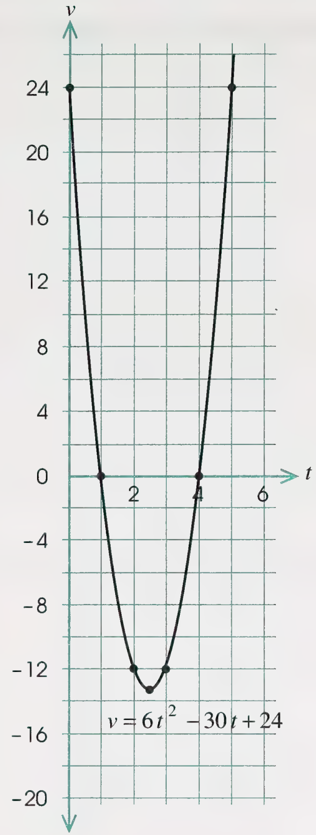
t	s
0	8
1	19
2	12
3	−1
4	−8
5	3



d. When $t = 0$, the direction of the motion is positive.
When $t = 1$, the object stops.
When $t = 3$, the direction of the motion is negative.
When $t = 4$, the object stops.
When $t = 5$, the direction of the motion is positive.

e.

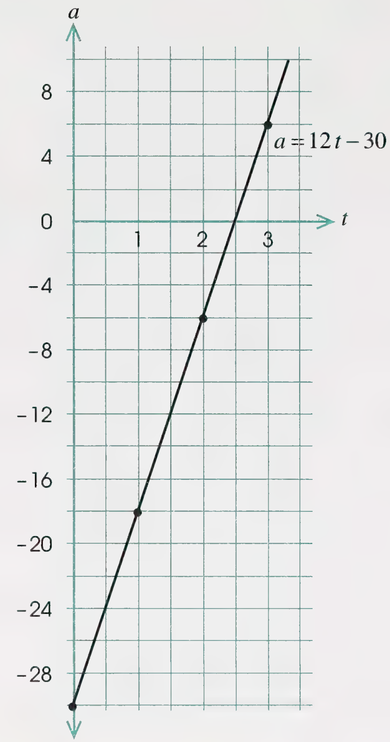
t	v
0	24
1	0
2	−12
$2\frac{1}{2}$	$-13\frac{1}{2}$
3	−12
4	0
5	24



The velocity is decreasing from $t = 0$ to $t = 2\frac{1}{2}$. At $t = 2\frac{1}{2}$, the velocity is a minimum. After $t = 2\frac{1}{2}$, the velocity is increasing.

f.

t	0	1	2	3
a	−30	−18	−6	6



The acceleration is increasing at a constant rate.

4. $s = 3t^3 - 3t^2 + t + 9$

$v = \frac{ds}{dt} = 9t^2 - 6t + 1$

$a = \frac{dv}{dt} = 18t - 6$

When $v = 4$ m/s, $\quad 9t^2 - 6t + 1 = 4$

$$9t^2 - 6t - 3 = 0$$

$$3\left(3t^2 - 2t - 1\right) = 0$$

$$3(t-1)(3t+1) = 0$$

$t - 1 = 0$ or $3t + 1 = 0$

$t = 1 \qquad t = -\frac{1}{3} \qquad \left(\text{Discard } t = -\frac{1}{3} \text{ since } t \geq 0.\right)$

When $t = 1$, $a = 18(1) - 6$

$a = 12$

When the velocity is 4 m/s, the acceleration is 12 m/s^2.

5. $s = \frac{t^2}{3+t}$

$$v = \frac{ds}{dt} = \frac{(3+t)\frac{d}{dt}\left(t^2\right) - t^2 \frac{d}{dt}(3+t)}{(3+t)^2}$$

$$= \frac{(3+t)(2t) - t^2(1)}{(3+t)^2}$$

$$= \frac{6t + 1t^2}{(3+t)^2}$$

If $v = \frac{3}{4}$ m/s, then $\frac{6t+t^2}{(3+t)^2} = \frac{3}{4}$.

$\left(6t + t^2\right)(4) = 3(3+t)^2$

$24t + 4t^2 = 3\left(9 + 6t + t^2\right)$

$24t + 4t^2 = 27 + 18t + 3t^2$

$t^2 + 6t - 27 = 0$

$(t-3)(t+9) = 0$

$t - 3 = 0$ or $t + 9 = 0$

$t = 3 \qquad t = -9 \qquad (\text{Discard } t = -9 \text{ since } t \geq 0.)$

$$\begin{aligned} a = \frac{dv}{dt} &= \frac{(3+t)^2 \frac{d}{dt}\left(6t+t^2\right)-\left(6t+t^2\right)\frac{d}{dt}(3+t)^2}{(3+t)^4} \\ &= \frac{(3+t)^2(6+2t)-\left(6t+t^2\right)(2)(3+t)}{(3+t)^4} \\ &= \frac{\left(9+6t+t^2\right)(6+2t)-\left(6t+t^2\right)(6+2t)}{(3+t)^4} \\ &= \frac{(6+2t)\left(9+6t+t^2-6t-t^2\right)}{(3+t)^4} \\ &= \frac{9(6+2t)}{(3+t)^4} \\ &= \frac{18(3+t)}{(3+t)^4} \\ &= \frac{18}{(3+t)^3} \end{aligned}$$

When $t=3$, $a = \frac{18}{(3+3)^3}$

$$\begin{aligned} &= \frac{18}{216} \\ &= \frac{1}{12} \end{aligned}$$

When the velocity is $\frac{3}{4}$ m/s, the acceleration is $\frac{1}{12}$ m/s^2.

6. **a.** positive **b.** negative **c.** positive
   d. zero **e.** negative **f.** positive

7. $v(t) = \frac{ds}{dt} = 6t^2 - 24$

   When $v=0$, $6t^2-24=0$

   $$\begin{aligned} t^2 &= 4 \\ t &= 2 \end{aligned}$$

   $$\begin{aligned} a(t) &= \frac{dv}{dt} = 12t \\ a(2) &= 12(2) \\ &= 24 \end{aligned}$$

   Therefore, the acceleration is 24 cm/s^2 when the velocity is 0 cm/s.

8. **a.** $s(t) = t^3 - 3t^2 - 12t$

   $v(t) = 3t^2 - 6t - 12$

   $a(t) = 6t - 6$

   When $a=0$, $6t-6=0$

   $t = 1$ s

   b. $s(1) = (1)^3 - 3(1)^2 - 12(1)$

   $= -14$ m

   $v(1) = 3(1)^2 - 6(1) - 12$

   $= -15$ m/s

c. The object is 14 m to the left of the origin and is travelling -15 m/s away from the origin at a steady velocity $\left(a = 0 \text{ m/s}^2\right)$.

9. $s(t) = t^4 - 10t^3 + 36t^2 + 10t + 12$

$v(t) = 4t^3 - 30t^2 + 72t + 10$

$$\begin{aligned} a(t) &= 12t^2 - 60t + 72 \\ &= t^2 - 5t + 6 \\ &= (t-2)(t-3) \end{aligned}$$

$a(t) > 0$ when $(t-2)(t-3) > 0$

$(t-2)(t-3) > 0$ when $t - 2 > 0$ and $t - 3 > 0$

$t > 2 \qquad t > 3$

Therefore, $t > 3$ s.

Also, $(t-2)(t-3) > 0$ when $t - 2 < 0$ and $t - 3 < 0$

$t < 2 \qquad t < 3$

Therefore, $t < 2$ s. Since time is always positive, $t < 2$ becomes $0 \text{ s} < t < 2$ s.

Thus, acceleration is positive when $t > 3$ s and $t < 2$ s $(t > 0 \text{ s})$; and acceleration is negative when $2 \text{ s} < t < 3$ s.

10. Velocity is negative because it is moving away and acceleration is also negative because it is in the same direction as the velocity.

11. The bus stops when velocity $= 0$.

$$\begin{aligned} -15t + 30 &= 0 \\ t &= 2 \end{aligned}$$

It takes the bus 2 s to stop.

12. $\frac{d}{dt}\left(3v^2\right) = \frac{d}{dt}(18s) + \frac{d}{dt}(300)$

$$6v\frac{dv}{dt} = 18\frac{ds}{dt} + 0$$

Substitute $\frac{dv}{dt} = a$ and $\frac{ds}{dt} = v$ into the equation.

$$\begin{aligned} 6va &= 18v \\ a &= \frac{18v}{6v} \\ &= 3 \end{aligned}$$

13. $\frac{d}{dt}(E) = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}(mps)$

$$\begin{aligned} 0 &= \frac{1}{2}m(2v)\frac{dv}{dt} + mp\frac{ds}{dt} \\ &= mva + mpv \\ mva &= -mpv \\ a &= -\frac{mpv}{mv} \\ &= -p \end{aligned}$$

14. $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{6000s}{6+s}\right)$

$$= \frac{(6+s)\frac{d}{dt}(6000s) - (6000s)\frac{d}{dt}(6+s)}{(6+s)^2}$$

$$= \frac{(6+s)(6000)\frac{ds}{dt} - (6000s)\frac{ds}{dt}}{(6+s)^2}$$

$$= \frac{36\,000v + 6000vs - 6000vs}{(6+s)^2}$$

$$= \frac{36\,000v}{(6+s)^2}$$

$$= \frac{36\,000\left(\frac{6000s}{6+s}\right)}{(6+s)^2}$$

$$= \frac{216\,000\,000s}{(6+s)^3}$$

15. $\frac{d}{dt}\left(\frac{1}{v}\right) + \frac{d}{dt}\left(\frac{1}{3}\right) = \frac{d}{dt}(4t)$

$$\left(-\frac{1}{v^2}\right)\frac{dv}{dt} + 0 = 4$$

$$-\frac{a}{v^2} = 4$$

$$-a = 4v^2$$

$$a = -4v^2$$

16. $\frac{d}{dx}\sqrt{y^2-2} = \frac{d}{dx}(xy)$

$$\frac{1}{2}\left(y^2-2\right)^{-\frac{1}{2}}\frac{d}{dx}\left(y^2-2\right) = x\frac{dy}{dx} + y$$

$$\frac{\cancel{2}y\frac{dy}{dx}}{\cancel{2}\left(y^2-2\right)^{\frac{1}{2}}} = x\frac{dy}{dx} + y$$

$$y\frac{dy}{dx} = x\left(y^2-2\right)^{\frac{1}{2}}\frac{dy}{dx} + y\left(y^2-2\right)^{\frac{1}{2}}$$

$$\left[y - x\left(y^2-2\right)^{\frac{1}{2}}\right]\frac{dy}{dx} = y\left(y^2-2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{y\left(y^2-2\right)^{\frac{1}{2}}}{y - x\left(y^2-2\right)^{\frac{1}{2}}}$$

17. $\frac{d}{dx}\left(x^3y\right) - \frac{d}{dx}\left(4y^2\right) = \frac{d}{dx}(3)$

$$x^3\frac{dy}{dx} + y\left(3x^2\right) - 8y\frac{dy}{dx} = 0$$

$$\left(x^3 - 8y\right)\frac{dy}{dx} = -3x^2y$$

$$\frac{dy}{dx} = \frac{-3x^2y}{x^3-8y} \text{ or } \frac{3x^2y}{8y-x^3}$$

Section 2: Activity 3

1. $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

If $d = 12$, then $r = 6$.

When $r = 6$, $\frac{dA}{dr} = 2\pi(6)$

$$= 12\pi \text{ cm}^2/\text{cm}$$

2. $V = e^3$

$$\frac{dV}{de} = 3e^2$$

When $e = 5$, $\frac{dV}{de} = 3(5)^2$

$$= 75 \text{ mm}^3/\text{mm}$$

3. $v = \frac{P}{4\eta\ell}\left(R^2 - r^2\right)$

$$\frac{dv}{dr} = \frac{-Pr}{2\eta\ell}$$

$$= \frac{-4000(0.004)}{2(0.027)(2)}$$

$$= -148.1 \text{ (cm/s)/cm}$$

Therefore, the velocity of bloodflow decreases by 148.1 cm/s for every centimetre from the centre of the artery.

4. $n = 250 + 10t + t^2$

$$\frac{dn}{dt} = 10 + 2t$$

When $t = 3$, $\frac{dn}{dt} = 10 + 2(3)$

$$= 16$$

The growth rate after three hours is 16 bacteria/h.

5. $C(x) = 5000 + 5x + 0.05x^2$

$$\frac{dC}{dx} = 5 + 0.1x$$

When $x = 100$, $\frac{dC}{dx} = 5 + 0.1(100)$

$$= \$15$$

When $x = 101$, $\frac{dC}{dx} = 5 + 0.1(101)$

$$= \$15.10$$

The marginal cost of producing 100 items is 10¢ less per item than the marginal cost of producing 101 items.

Section 2: Follow-up Activities

Extra Help

1. a. $v = \frac{ds}{dt} = 6t^2 - 42t + 60$

When $t = 2$,
$$\begin{aligned} v &= 6(2)^2 - 42(2) + 60 \\ &= 24 - 84 + 60 \\ &= 0 \end{aligned}$$

Remember that s is a maximum or minimum when $v = 0$. When $a = 0$, v is a maximum or minimum.

b. $a = \frac{dv}{dt} = 12t - 42$

When $t = 3$,
$$\begin{aligned} a &= 12(3) - 42 \\ &= 36 - 42 \\ &= -6 \end{aligned}$$

c. When $t = 1$,
$$\begin{aligned} s &= 2(1)^3 - 21(1)^2 + 60(1) + 80 \\ &= 2 - 21 + 60 + 80 \\ &= 121 \end{aligned}$$

When $t = 3$,
$$\begin{aligned} s &= 2(3)^3 - 21(3)^2 + 60(3) + 80 \\ &= 54 - 189 + 180 + 80 \\ &= 125 \end{aligned}$$

$$\begin{aligned} v_{ave} &= \frac{125 - 121}{3 - 1} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

The average velocity is 2 m/s.

d. When $a = 0$, v is a maximum or minimum.

$$\begin{aligned} 12t - 42 &= 0 \\ 12t &= 42 \\ t &= \frac{42}{12} \\ &= 3.5 \end{aligned}$$

When $t = 3.5$,
$$\begin{aligned} v &= 6(3.5)^2 - 42(3.5) + 60 \\ &= 73.5 - 147 + 60 \\ &= -13.5 \end{aligned}$$

If the coefficient of the second-degree term in a quadratic function is positive, then the function has a minimum value. If the coefficient of the second-degree term in a quadratic function is negative, then the function has a maximum value.

The minimum velocity is -13.5 m/s.

$v = 6t^2 - 42t + 60$ (The coefficient of t^2 is positive; thus, velocity is a minimum.)

2. $x^2 + 2xy + 4y^2 = 3$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(2xy) + \frac{d}{dx}\left(4y^2\right) = \frac{d}{dx}(3)$$

$$2x + 2x\frac{dy}{dx} + 2y + 8y\frac{dy}{dx} = 0$$

$$(2x + 8y)\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 8y}$$

$$= \frac{-x - y}{x + 4y}$$

3. $\frac{d}{dt}\left(3v^2\right) = \frac{d}{dt}(5s) + \frac{d}{dt}(1000)$

$$6v\frac{dv}{dt} = 5\frac{ds}{dt} + 0$$

$$6va = 5v$$

$$a = \frac{5v}{6v}$$

$$= \frac{5}{6}$$

The acceleration at any time is $\frac{5}{6}$.

Enrichment

1. $\frac{d}{dt}\left(s^2\right) + 25\frac{d}{dt}\left(v^2\right) = 0$

$$2s\frac{ds}{dt} + 50v\frac{dv}{dt} = 0$$

Substitute $\frac{ds}{dt} = v$ and $\frac{dv}{dt} = a$ into the equation.

$$2sv + 50va = 0$$

$$s = \frac{-50va}{2v}$$

$$s = -25a$$

2. $\frac{dv}{dt} = \frac{\left(3s^2 - 2\right)\frac{d}{dt}\left(1 - 3s^2\right) - \left(1 - 3s^2\right)\frac{d}{dt}\left(3s^2 - 2\right)}{\left(3s^2 - 2\right)^2}$

$$\frac{dv}{dt} = \frac{\left(3s^2 - 2\right)\left(-6s\frac{ds}{dt}\right) - \left(1 - 3s^2\right)6s\frac{ds}{dt}}{\left(3s^2 - 2\right)^2}$$

Substitute $\frac{ds}{dt} = v$ and $\frac{dv}{dt} = a$ into the equation, and factor.

$$a = \frac{(3s^2-2)(-6sv)-(1-3s^2)(6sv)}{(3s^2-2)^2}$$

$$= \frac{-6sv(3s^2-2-3s^2+1)}{(3s^2-2)^2}$$

$$= \frac{6sv}{(3s^2-2)^2}$$

$$v = \frac{1-3s^2}{3s^2-2} \quad \text{(from the original equation)}$$

$$a = \frac{6s\left(\frac{1-3s^2}{3s^2-2}\right)}{(3s^2-2)^2}$$

$$= \frac{6s(1-3s^2)}{(3s^2-2)^3}$$

Section 3: Activity 1

1. $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Substitute $\frac{dA}{dt} = \frac{\pi}{2}\ \text{cm}^2/\text{min}$ into the equation.

$$\frac{\pi}{2} = 2\pi r \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{\cancel{\pi}}{2(2)\cancel{\pi} r}$$

$$= \frac{1}{4r}$$

Determine the radius of the circle when the area of the circle is $9\pi\ \text{cm}^2$.

$$A = \pi r^2$$

$$r = \sqrt{\frac{A}{\pi}}$$

$$= \sqrt{\frac{9\cancel{\pi}}{\cancel{\pi}}}$$

$$= 3$$

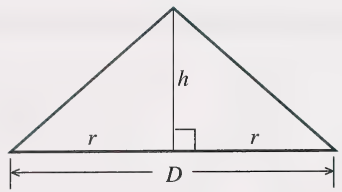
$$\therefore \frac{dr}{dt} = \frac{1}{4r}$$

$$= \frac{1}{4(3)}$$

$$= \frac{1}{12}$$

The radius of the circle is increasing at the rate of $\frac{1}{12}$ cm/min.

2.



$$\frac{dV}{dt} = 8 \text{ m}^3/\text{min} \qquad (\text{given})$$

$$D = 2r = 3h$$

$$\therefore\ r = \frac{3}{2}h$$

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h \\ &= \frac{3}{4}\pi h^3 \end{aligned}$$

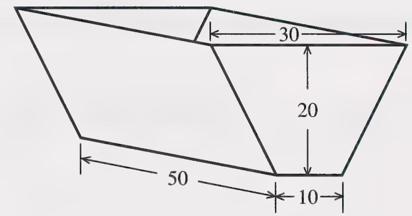
$$\begin{aligned} \frac{dV}{dt} &= \frac{3}{4}\pi\left(3h^2\right)\frac{dh}{dt} \\ 8 &= \frac{3}{4}\pi\left(3h^2\right)\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{32}{9\pi h^2} \end{aligned}$$

When $h = 4$, $\frac{dh}{dt} = \frac{32}{9\pi(4)^2}$

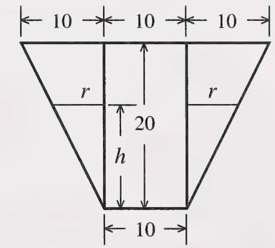
$$= \frac{2}{9\pi}$$

The height is increasing at $\frac{2}{9\pi}$ m/min.

3.



Area of a trapezoid $= \frac{1}{2}(a+b)h$



$$\frac{r}{h} = \frac{10}{20}$$

$$\therefore\ r = \frac{1}{2}h$$

Determine the area of the trapezoid when the height is h cm.

$$\begin{aligned} A &= \frac{1}{2}[10 + (2r + 10)]h \\ &= \frac{1}{2}(20 + 2r)h \\ &= (10 + r)h \\ &= \left(10 + \frac{h}{2}\right)h \qquad \left(\text{Substitute } r = \tfrac{1}{2}h.\right) \\ &= 10h + \frac{h^2}{2} \end{aligned}$$

Volume = Area of trapezoid × length

$$V = \left(10h + \frac{h^2}{2}\right) \times 50$$
$$= 500h + 25h^2$$

$$\frac{dV}{dt} = 500\frac{dh}{dt} + 50h\frac{dh}{dt}$$
$$= (500 + 50h)\frac{dh}{dt}$$

Substitute $\frac{dV}{dt} = 600\ \text{cm}^3/\text{s}$ into the equation.

$$600 = (500 + 50h)\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{600}{500 + 50h}$$

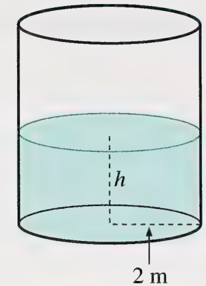
When $h = 10$, $\frac{dh}{dt} = \frac{600}{500 + 500}$

$$= \frac{600}{1000}$$
$$= 0.6$$

The water is rising at 0.6 cm/s.

4. $V = \pi r^2 h$

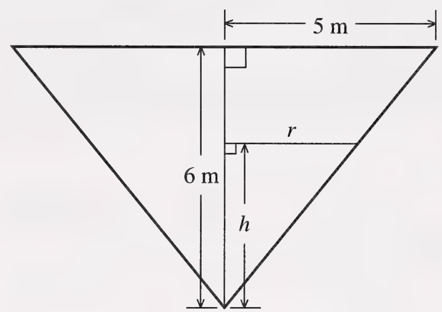
$$= \pi(2)^2 h$$
$$= 4\pi h$$



$$\frac{dV}{dt} = 4\pi\frac{dh}{dt} \quad \left(\frac{dV}{dt} = 3\ \text{m}^3/\text{min}\right)$$
$$\frac{dh}{dt} = \frac{3}{4\pi}$$

The water level is rising at a constant rate of $\frac{3}{4\pi}$ m/min. The depth of 4 m is not needed because it does not affect $\frac{dh}{dt}$.

5.



$$\frac{r}{h} = \frac{5}{6}$$
$$\therefore\ r = \frac{5}{6}h$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(\frac{5}{6}h\right)^2 h$$
$$= \frac{1}{3} \times \frac{25}{36}\pi h^3$$
$$= \frac{25}{108}\pi h^3$$

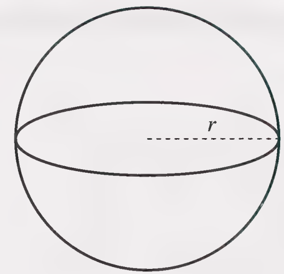
$$\frac{dV}{dt} = \frac{25}{108}\pi\left(3h^2\right)\frac{dh}{dt} \quad \left(\frac{dV}{dt} = 3\pi \text{ m}^3/\text{min}\right)$$

When $h = 3$, $3\pi = \frac{75}{108}\pi(3)^2 \frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{\overset{1}{\cancel{3}}\cancel{\pi}(\overset{12}{\cancel{108}})}{\underset{25}{\cancel{75}}\cancel{\pi}(\underset{1}{\cancel{9}})}$$
$$= \frac{12}{25}$$

The water level is rising $\frac{12}{25}$ m/min.

6.



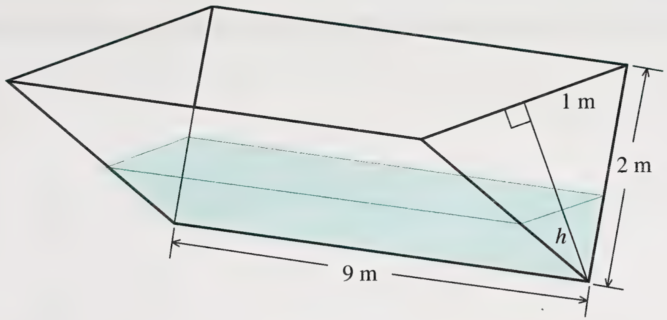
$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}$$

Substitute $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$ and $r = 5$ cm into the equation.

$$100 = 4\pi(5)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{100}{100\pi}$$
$$= \frac{1}{\pi}$$

The radius is increasing at $\frac{1}{\pi}$ cm/s.

7.



Height of the triangle $= \sqrt{2^2 - 1^2}$ (Phythagorean property)

$= \sqrt{3}$

$$\frac{\sqrt{3}}{1} = \frac{h}{r}$$

$$\therefore\ r = \frac{1}{\sqrt{3}}h$$

$V =$ Area of triangular end $\times$ length

$$= \frac{1}{2}(2r)h \times 9$$

$$= 9rh$$

$$= \left(\frac{9}{\sqrt{3}}h\right)h$$

$$= \frac{9}{\sqrt{3}}h^2$$

$$\frac{dV}{dt} = \frac{9}{\sqrt{3}}(2h)\frac{dh}{dt}$$

$\frac{dV}{dt} = 3\ \text{m}^3/\text{min}$ and $h = 0.5$ m

$$3 = \frac{9}{\sqrt{3}}\left(2 \times \frac{1}{2}\right)\frac{dh}{dt}$$

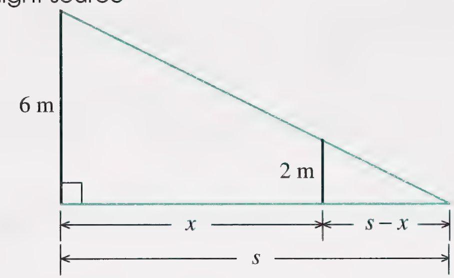
$$\frac{dh}{dt} = \frac{3\sqrt{3}}{9}$$

$$= \frac{\sqrt{3}}{3}$$

The water level is rising at $\frac{\sqrt{3}}{3}$ m/min.

Section 3: Activity 2

1. light source



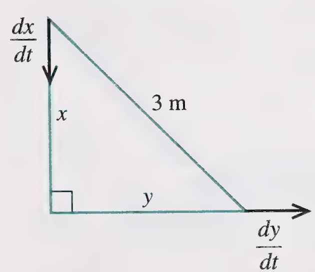
$\frac{dx}{dt} = 1.5$ m/s (given)

In the preceding diagram, the two triangles are similar and their corresponding sides are proportional.

$$\frac{2}{6} = \frac{s-x}{s}$$
$$2s = 6s - 6x$$
$$\frac{d}{dt}(2s) = \frac{d}{dt}(6s) - \frac{d}{dt}(6x)$$
$$2 \bullet \frac{ds}{dt} = 6 \bullet \frac{ds}{dt} - 6 \bullet \frac{dx}{dt}$$
$$4 \bullet \frac{ds}{dt} = 6 \bullet \frac{dx}{dt}$$
$$\frac{ds}{dt} = \frac{3}{2} \bullet \frac{dx}{dt}$$
$$= \frac{3}{2}(1.5)$$
$$= 2.25$$

The end of the shadow of the post is moving at 2.25 m/s.

2.



Given $\frac{dx}{dt} = -0.5$ m/s, find $\frac{dy}{dt}$ when $y = 2$.

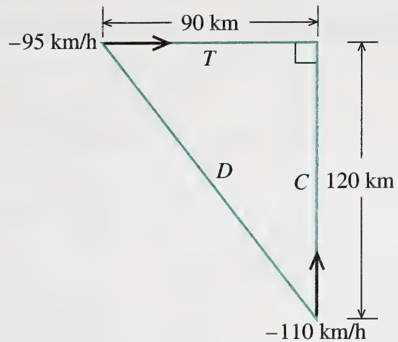
$$x^2 + y^2 = 3^2$$
$$2x \bullet \frac{dx}{dt} + 2y \bullet \frac{dy}{dt} = 0$$
$$2y \bullet \frac{dy}{dt} = -2x \bullet \frac{dx}{dt}$$
$$\frac{dy}{dt} = \frac{-x}{y} \bullet \frac{dx}{dt}$$

When $y = 2,\ x^2 + 2^2 = 3^2$
$$x^2 = 9 - 4$$
$$x = \sqrt{5}$$

$$\therefore \frac{dy}{dt} = \frac{-\sqrt{5}(-0.5)}{2}$$
$$= \frac{\sqrt{5}}{4}$$

The post is sliding away from the base of the wall at a rate of $\frac{\sqrt{5}}{4}$ m/s (approximately 0.56 m/s).

3.



The speeds of the vehicles are negatives because distances C and T are decreasing.

Given $\frac{dT}{dt} = -95$ km/h and $\frac{dC}{dt} = -110$ km/h, find $\frac{dD}{dt}$ when $T = 90$ km and $C = 120$ km.

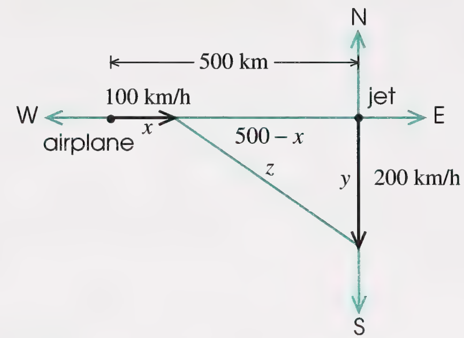
$$D^2 = T^2 + C^2$$
$$D^2 = 90^2 + 120^2$$
$$D = 150$$

$$D^2 = T^2 + C^2$$

$$2D \bullet \frac{dD}{dt} = 2T \bullet \frac{dT}{dt} + 2C \bullet \frac{dC}{dt}$$
$$2(150)\frac{dD}{dt} = 2(90)(-95) + 2(120)(-110)$$
$$\frac{dD}{dt} = \frac{-17\,100 - 26\,400}{300}$$
$$= -145$$

The distance between the vehicles is decreasing at a rate of 145 km/h.

4.



Let x be the distance the plane has travelled, and y be the distance the jet has travelled.

Fig 108

Given $\frac{dx}{dt} = 100$ km/h and $\frac{dy}{dt} = 200$ km/h, find $\frac{dz}{dt}$ when $t = 2$ h.

After 2 h, $x = 2(100)$

$= 200$

After 2 h, $y = 2(200)$

$= 400$

$$\begin{aligned} z^2 &= (500 - x)^2 + y^2 \\ z^2 &= (500 - 200)^2 + 400^2 \\ z^2 &= 300^2 + 400^2 \\ z &= 500 \end{aligned}$$

$$\begin{aligned} z^2 &= (500 - x)^2 + y^2 \\ &= 250\,000 - 1000x + x^2 + y^2 \end{aligned}$$

$$2z \bullet \frac{dz}{dt} = -1000 \bullet \frac{dx}{dt} + 2x \bullet \frac{dx}{dt} + 2y \bullet \frac{dy}{dt}$$

$$\begin{aligned} 2(500)\frac{dz}{dt} &= -1000(100) + 2(200)(100) + 2(400)(200) \\ \frac{dz}{dt} &= \frac{-100\,000 + 40\,000 + 160\,000}{1000} \\ &= 100 \end{aligned}$$

The distance between the plane and the jet is increasing at 100 km/h.

5. $\dfrac{dy}{dt} = 4x \bullet \dfrac{dx}{dt}$

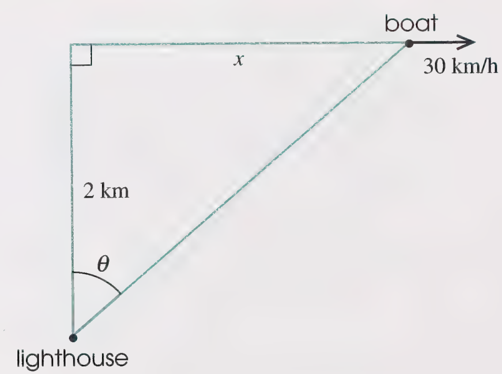
Substitute $\frac{dx}{dt} = 4$ and $x = 3$ into the equation.

$$\begin{aligned} \frac{dy}{dt} &= 4(3)(4) \\ &= 48 \end{aligned}$$

When $x = 3$, y is changing at 48 units/s.

Section 3: Activity 3

1.



Let θ be the angle swept out by the beam.

Let x be the distance the boat has sailed from its initial position, opposite the lighthouse.

Given $\frac{dx}{dt} = 30$ km/h, find $\frac{d\theta}{dt}$ when $x = 4$ km.

$$\tan\theta = \frac{x}{2}$$

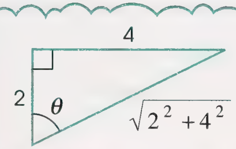
$$\sec^2\theta \bullet \frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{2} \bullet \frac{dx}{dt}$$

When $x = 4$, $\cos\theta = \frac{2}{\sqrt{2^2+4^2}}$.

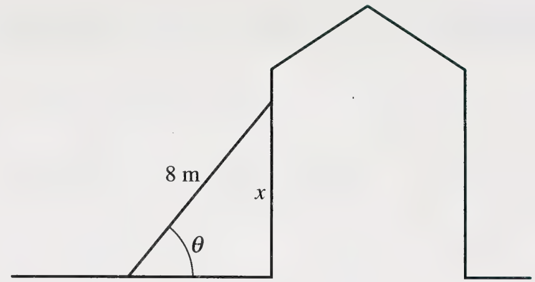
$$\therefore \frac{d\theta}{dt} = \left(\frac{1}{2}\right)(30)\left(\frac{2}{\sqrt{2^2+4^2}}\right)^2$$
$$= 15\left(\frac{4}{4+16}\right)$$
$$= \frac{60}{20}$$
$$= 3$$

The beam is revolving at a rate of 3 rad/h.



Use the Pythagorean theorem.

2.



Let θ be the angle the ladder makes with the ground.

Given $\frac{d\theta}{dt} = \frac{-1}{4}$, find $\frac{dx}{dt}$ when $\theta = \frac{\pi}{4}$.

$$\sin\theta = \frac{x}{8}$$

$$\cos\theta \bullet \frac{d\theta}{dt} = \frac{1}{8} \bullet \frac{dx}{dt}$$

$$\frac{dx}{dt} = 8\cos\theta \bullet \frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{4}$, $\cos\theta = \frac{\sqrt{2}}{2}$.

Recall that $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ from the unit circle.

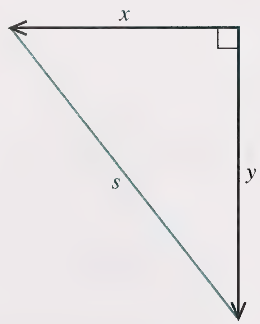
$$\frac{dx}{dt} = 8 \bullet \frac{\sqrt{2}}{2} \bullet \frac{-1}{4}$$
$$= -\sqrt{2}$$

The ladder is sliding down the wall at $\sqrt{2}$ m/s.

Section 3: Follow-up Activities

Extra Help

1.



After 2 h, $x = 2 \times 50 = 100$ and $y = 2 \times 80 = 160$.

$$\therefore\ s^2 = x^2 + y^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Since $\frac{dx}{dt} = 50$ km/h, $\frac{dy}{dt} = 80$ km/h.

If $x = 100$ and $y = 160$, then $s \doteq 188.68$.

$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$188.68\frac{ds}{dt} \doteq 100(50) + 160(80)$$

$$\frac{ds}{dt} \doteq \frac{17\,800}{188.68}$$

$$\doteq 94.34$$

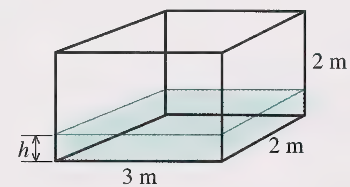
They are separating at approximately 94.34 km/h after 2 h.

2. Find the volume when the water is h m deep.

$$V = \ell \times w \times h$$

$$= 3 \times 2 \times h$$

$$= 6h$$



$$\frac{dV}{dt} = 6\frac{dh}{dt}$$

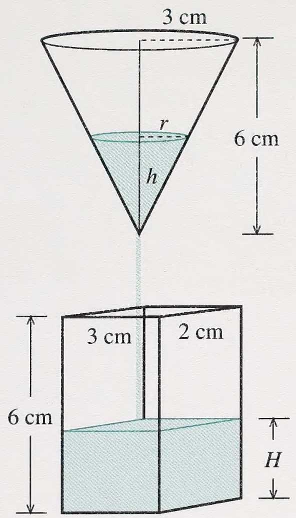
Since $\frac{dV}{dt} = 1.5\ \text{m}^3/\text{min}$, $1.5 = \frac{6\,dh}{dt}$

$$\frac{dh}{dt} = \frac{1.5}{6}$$

$$= 0.25$$

The water is rising at 0.25 m/min.

Enrichment



Volume of a cone $V = \frac{1}{3}\pi r^2 h$

$$\frac{r}{h} = \frac{3}{6}$$

$$r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$= \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}\left(3h^2\right)\frac{dh}{dt}$$

$$= \frac{9\pi}{4}\left(\frac{1}{2}\right) \qquad \left(\text{At } h = 3 \text{ cm}, \ \frac{dh}{dt} = \frac{1}{2} \text{ cm/s.}\right)$$

$$= \frac{9\pi}{8} \text{ cm}^3/\text{s} \qquad (\text{the rate of flow out of the filter})$$

For the rectangular container, $V = \ell \times w \times H$

$$= 3 \times 2 \times H$$

$$= 6h$$

$$\frac{dV}{dt} = 6\frac{dH}{dt}$$

$$\frac{9\pi}{8} = \frac{6\,dH}{dt}$$

$$\frac{dH}{dt} = \frac{9\pi}{8 \times 6}$$

$$= \frac{3\pi}{16}$$

The water level in the rectangular container is rising $\frac{3\pi}{16}$ cm/s.

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